Tuning of PID Controllers Based on Sensitivity Margin Specification

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Abstract

This paper presents an alternative method to the one proposed by Åström and Hägglund (1995) for the tuning of PID controllers with specification of sensitivity margin M_s. The method is based on the following steps: 1º) to delimit the range of frequencies in which it is possible to contact with a specific point of a circle of radius M_s^{-1} centered at -1, 2º) to explore the frequencies where this contact takes place tangentially, because these frequencies will only be possible solutions, and 3º) to choose one of the possible solutions. Some examples have been selected in order to show the main features of the proposed method.

1 Introduction

In PID control design, one often needs to go beyond the issue of closed loop stability. In particular, it is common to specify some quantitative measures of how far from instability the nominal loop is. This is attained by introducing measures that describe the distance from the nominal open-loop frequency response to the critical stability point (-1;0).

Two indicators of relative stability for the case in which the open loop has no poles in the open RHP are the *gain margin*, A_m, and the *phase margin*, ϕ_m. Figure 1 shows an alternative measure for relative stability. The radius of the circle tangent to the polar plot of $L(j\omega) = G_p(j\omega)G_c(j\omega)$ is the reciprocal of the *nominal sensitivity peak* M_s. The larger the sensitivity peak is, the closer the loop will be to instability.

The sensitivity peak is a more reliable indicator of relative stability than the gain and phase margin. It is possible to find situations where the gain and phase margin are good, yet a very large sensitivity peak informs us of a very subtle stability condition. The reverse is not true: sensitivity performance gives always minimal value for the gain and phase margins.

Minimum bounds on both the gain margin and the phase margin of a system characterized by a minimum phase $L(j\omega)$ can be expressed directly as function of M_s^{-1}, as shown in Figure 1. Note that any loop gain point will lie outside the circle of radius M_s^{-1} so that $|L(j\omega_{cg})| = A_m^{-1} \le d$. Therefore, the distance $d = 1-M_s^{-1}$ shown in Figure 1 represents a lower bound on the gain margin of the system in the sense that

$$A_m \ge \frac{1}{d} = \frac{M_s - 1}{M_s} \qquad (1)$$

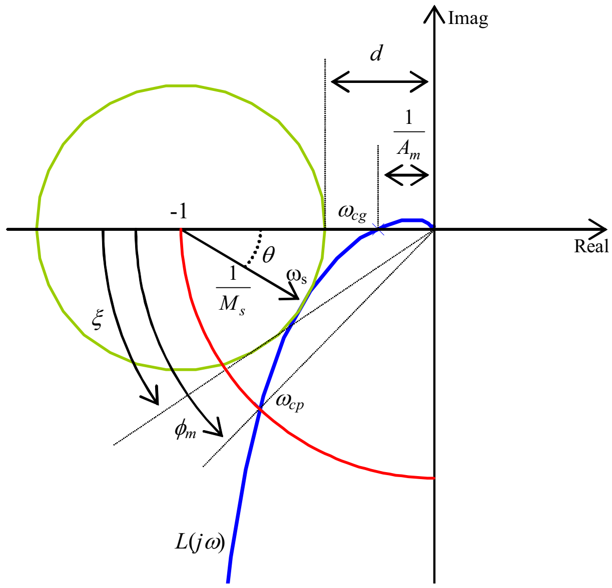

Figure 1: Open loop gain and M_s^{-1} circle

Furthermore, the angle ξ depicted in Figure 1 represents a lower bound on the phase margin of the system in the sense that

$$\phi_m \ge \xi = 2\,sin^{-1}\left(\frac{1}{M_s}\right) \qquad (2)$$

To ensure, for example, an acceptable nominal design that attains $A_m \ge 2$ and $\phi_m \ge 30º$ we may now require that

$$M_s \le 2 \cong 6\,dB \;\; or\, that \;\; M_s^{-1} = \min_{\omega} |1 + L(j\omega)| \ge 0.5 \qquad (3)$$

so that $A_m \ge 2$ and $\phi_m \ge 29º \cong 30º$, in view of (1) and (2), respectively. However, such a requirement may be conservative, since larger values of M_s^{-1} can imply both $A_m > 2$ and $\phi_m > 30^o$.

Note that the single sensitivity function requirement defined by (3) can be used to replace both the A_m and the ϕ_m requirements not only in the minimum phase cases but also in the unstable and non minimum phase cases as well. In particular, (3) will ensure that $L(j\omega)$ remains an acceptable, "marginal" distance away from the critical -1 point, irrespective of the number and the direction of encirclements required for closed loop stability; that is, (3) will ensure robust stability with respect to plant parameter variations once nominal closed-loop stability has been obtained.

The paper is organized as follows. Section 2 is devoted to introduce basic concepts of tuning of PID controllers in the frequency domain. In section 3 the tuning of PID controller when the design criterion is to obtain a specified sensitivity M_s is explained in a detailed way. Some illustrative examples are described in section 4. Finally section 5 presents some conclusions.

2 Tuning of PID in the frequency domain

Some frequency domain tuning methods of PID controllers (Åström and Hägglund (1995), Morilla and Dormido (2000)) can be interpreted in terms of moving one point A in the frequency response of the process $G_p(j\omega)$ to an arbitrary point B. This point B is the specification or is determined by the specification (phase margin, gain margin). Therefore, once point B has been fixed there is a range of frequencies in the Nyquist plot of $G_p(j\omega)$ where it is possible to find a solution. This range can be determined from the following angular condition:

$$\phi_c^{\min} \leq (\phi_B - \phi(\omega)) \leq \phi_c^{\max} \quad (4)$$

where ϕ_B and $\phi(\omega)$ are the phase of point B and $G_p(j\omega)$, with respect to the negative real axis, and $\phi_c^{\min}$ and $\phi_c^{\max}$ are the minimum and maximum angular contributions of the controller. The range of frequencies depends therefore on the type of controller to be tuned.

Figure 2 shows an example, where the controller is a PID controller. Point B belongs to the bisector of the third quadrant, and the possible points A are represented in continuous trace (ω^{min} and ω^{max} delimitate the corresponding range of frequencies). The example includes as a particular case a specification of phase margin equal to 45º (point B is located on the unit circle).

Another important consideration in Morilla and Dormido (2000) was the convenience of using an auxiliary criterion (the maximum integral gain) together with the specification of phase or gain margin to choose the design frequency among all the possible solutions. However this constraint is not necessary when the tuning is made by combining both specifications.

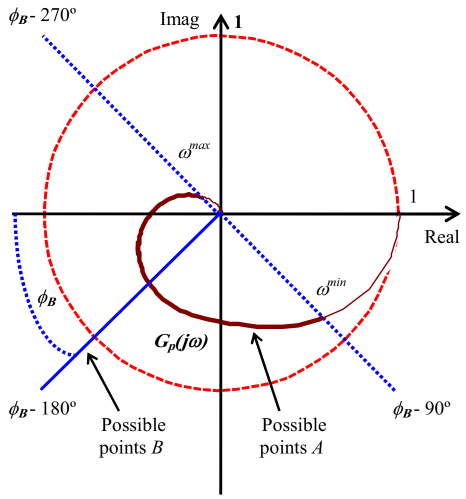

Figure 2: Range for the design frequency

Figure 3 shows the interpretation of the tuning by combined phase-gain margin. The tuning procedure can be seen as the change, in the complex plane, of two points (A and D) on $G_p(j\omega)$ into points (B and E) on $L(j\omega)$. Figure 4 shows the interpretation of this tuning procedure in the parametric plane K_P (proportional gain) - K_I (integral gain), (Shafiei and Shenton, 1997).

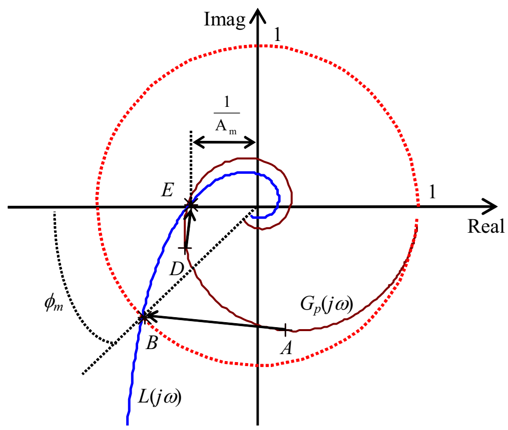

Figure 3: PID tuning by combined phase-gain margin

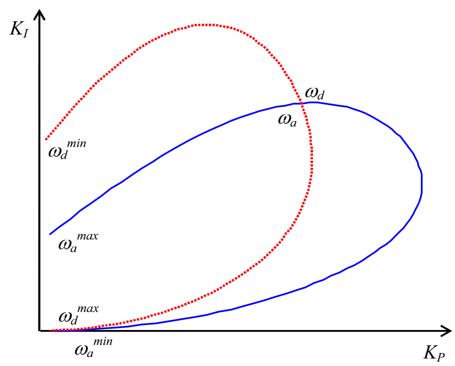

Figure 4: PID tuning in the parametric plane

3 Tuning of PID with specification of M_s.

In (Aström and Hägglund (1995)) the design criterion is to obtain a specified sensitivity M_s (see Figure 5) with good rejection of load disturbances. From the numerical point of view the procedure is reduced to the resolution of a system of three non-linear equations with five unknowns: the three PID controller parameters K_P, K_I and K_D, the frequency ω in which the specified sensitivity will take place and the angle θ where the contact with the circle M_s will happen.

For the resolution of these equations they propose to use an optimization procedure, combined either with the maximum integral gain criterion or the maximum frequency; the value of θ is bounded to the interval [0 π/2].

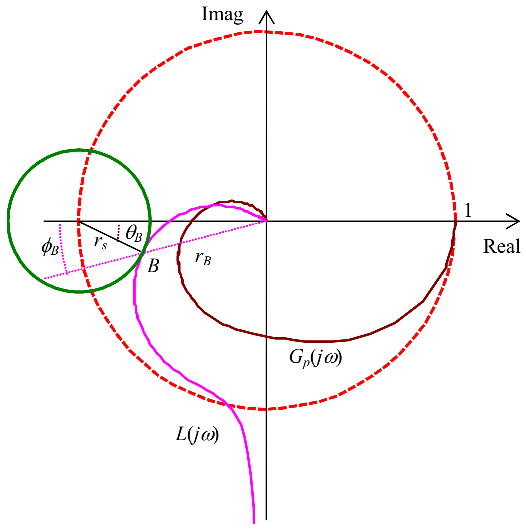

Figure 5: Nyquist plot of $G_p(j\omega)$ and $L(j\omega)$

The system of equations, with certain changes with respect to the formulation of Aström and Hägglund (1995), is the following:

$$\left[-K_P\ cos\phi(\omega)-\left(\frac{K_I}{\omega}+K_D\,\omega\right)sin\phi(\omega)\right]r(\omega)=-1+\frac{cos\theta}{M_s} \quad (5)$$

$$\left[-\left(K_P-K_D\,\omega\right)sin\phi(\omega)+\frac{K_I}{\omega}cos\phi(\omega)\right]r(\omega)=-\frac{sin\theta}{M_s} \quad (6)$$

$$a(\omega,\theta)K_P+b(\omega,\theta)K_I+c(\omega,\theta)K_D=0 \quad (7)$$

(5) and (6) equations impose the contact of $L(j\omega)$ with the circle M_s, with an angle θ, while (7) imposes the tangency condition.

$r(\omega)$ and $\phi(\omega)$ are the magnitude and the phase of $G_p(j\omega)$, $a(\omega,\theta)$, $b(\omega,\theta)$ and $c(\omega,\theta)$ are three functions used to abbreviate equation (7) that depend on the derivatives $r'(\omega)$, $\phi'(\omega)$ of the magnitude and phase of $G_p(j\omega)$ as shown in the following expressions:

$$a(\omega,\theta)=\phi'(\omega)-\frac{r'(\omega)}{r(\omega)}cotan(\theta+\phi(\omega))$$

$$b(\omega,\theta)=\frac{1}{\omega^2}-\frac{r'(\omega)}{\omega\,r(\omega)}-\frac{\phi'(\omega)cotan(\theta+\phi(\omega))}{\omega}$$

$$c(\omega,\theta)=1+\frac{\omega\,r'(\omega)}{r(\omega)}+\omega\,\phi'(\omega)cotan(\theta+\phi(\omega))$$

3.1 New tuning procedure of PID controllers

The method for tuning consists of the following steps:

a) To specify exactly the point where the contact with the circle M_s will take place.

b) To determine the family of controllers able to make $L(j\omega)$ pass through the specified point.

c) To use the tangency condition (the proposal given by Aström and Hägglund (1995) or a simpler one given in this paper), to determine those controllers that guarantee the tangential contact with the specified point on circle M_s.

d) To choose one of the possible controllers.

Next it is described how the contact point can be specified, how the family of controllers can be determined and represented, and how the tangency condition can be formulated. Other aspects, related with practical application of the tuning method and the choice of the controllers, are discussed in the next section together with the different steps of the proposed algorithm.

Point B can be specified completely if specification M_s is accompanied by angle θ_B in which the contact will take place (see Figure 5). Point B is perfectly located in the complex plane by means of its real and imaginary part (X_B, Y_B) or its magnitude and phase (r_B, ϕ_B) whose dependence with respect to $r_s = M_s^{-1}$ and θ_B is given in the following expressions

$$X_B = -1 + r_s\,cos\,\theta_B = -\,r_B\,cos\,\phi_B$$

$$Y_B = -\,r_s\,sen\,\theta_B = -\,r_B\,sen\,\phi_B$$

$$tg\,\phi_B = \frac{r_s\,sen\,\theta_B}{r_s\,cos\,\theta_B - 1} \quad (9)$$

$$r_B = \frac{r_s\,sen\,\theta_B}{sen\,\phi_B} \quad (10)$$

The parameters of the PID controllers that are able to make $L(j\omega)$ pass through point B can be obtained by applying the following expressions (Morilla and Dormido, 2000):

$$K_P(\omega)=\frac{r_B\,cos(\phi_B-\phi(\omega))}{r(\omega)} \quad (11)$$

$$T_I(\omega)=\frac{1}{2\,\alpha\,\omega}\left(tg(\phi_B-\phi(\omega))+\sqrt{4\,\alpha+tg^2(\phi_B-\phi(\omega))}\right) \quad (12)$$

$$T_D(\omega) = \alpha\ T_I(\omega) \quad (13)$$

in the range of frequencies that verify the angular condition (4). Equation (9) admits this other equivalent expression:

$$tg\ \theta_B = \frac{r_B\ sen\ \phi_B}{1 - r_B\ cos\ \phi_B} \quad (14)$$

The tangent to circle M_s in a point (X_s, Y_s) is a function of the angle θ that can be obtained as follows:

$$\frac{d\,Y_s}{d\,X_s} = \frac{\dfrac{d\,(-r_s\ sen\,\theta)}{d\,\theta}}{\dfrac{d\,(-1 + r_s\ cos\,\theta)}{d\,\theta}} = \frac{-r_s\ cos\,\theta}{-r_s\ sen\,\theta} = \frac{1}{tg\,\theta}$$

The tangent to $L(j\omega)$ is a function of the frequency ω that can be obtained as follows:

$$\frac{d\ Imag[L(j\omega)]}{d\ Real[L(j\omega)]} = \frac{-\dfrac{d\,r_L(\omega)}{d\,\omega}\,sen\,\phi_L(\omega) - r_L(\omega)\dfrac{d\,\phi_L(\omega)}{d\,\omega}\,cos\,\phi_L(\omega)}{-\dfrac{d\,r_L(\omega)}{d\,\omega}\,cos\,\phi_L(\omega) + r_L(\omega)\dfrac{d\,\phi_L(\omega)}{d\,\omega}\,sen\,\phi_L(\omega)}$$

The equality of both tangents in the point B give us the following expression for θ_B:

$$tg\ \theta_B = \frac{r'_B(\omega)\,cos\ \phi_B - r_B\ \phi'_B(\omega)\,sen\ \phi_B}{r'_B(\omega)\,sen\ \phi_B + r_B\ \phi'_B(\omega)\,cos\ \phi_B} \quad (15)$$

where $r'_B(\omega)$ and $\phi'_B(\omega)$ represent the value of the derivative of the magnitude and phase of $L(j\omega)$ in the point B respectively.

Equaling equations (14) and (15) gives the following condition that must fulfill the magnitude and phase of point B and the respective derivatives of $L(j\omega)$ in point B.

$$r_B\ r'_B(\omega) = r'_B(\omega)\,cos\ \phi_B - r_B\ \phi'_B(\omega)\,sen\ \phi_B \quad (16)$$

3.2 Tuning algorithm of PID controllers

Given the plant $G_p(j\omega)$, the PID parameters can be tuned to meet a specified M_s in the following way:

Step 1: Initialization. The user specifies the value of M_s and the angle θ_B in order to locate point B and chooses the controller's type, PI or PID. In the last case specifies α ($\alpha = T_D/T_I$).

Step 2: Calculate ϕ_B and r_B by using (9) and (10).

Step 3: Calculate the range of frequencies, using the expression (4), where $L(j\omega)$ may pass through point B.

Step 4: Sweep the whole range of frequencies in order to determine in which frequencies the tangency condition is fulfilled. This sweeping makes the following calculations for all and each one of the frequencies:

- Calculate the control parameters by means of (11), (12) and (13).
- Calculate the magnitude $r_L(\omega)$ and phase $\phi_L(\omega)$ of $L(j\omega)$ in the vicinity of the design frequency, in particular in the five points ω-2Δω, ω-Δω, ω, ω+Δω y ω+2Δω, due to the following approximation for the calculation of the derivative $r_L(\omega)$ and $\phi_L(\omega)$ in point B (Savitzky and Golay, 1964)

$$\frac{d\,f(\omega)}{d\omega} \cong \frac{f(\omega-2\Delta\omega) - 8\,f(\omega-\Delta\omega) + 8\,f(\omega-\Delta\omega) - f(\omega+2\Delta\omega)}{12\Delta\omega} \quad (17)$$

it is also convenient to make sure that $r_L(\omega)$ and $\phi_L(\omega)$ coincide with values r_B and ϕ_B determined in Step 2.

- Calculate the derivatives $r'_B(\omega)$ and $\phi'_B(\omega)$, using the generic approximation (17)
- Evaluate the tangency condition, using the following quadratic function:

$$FCT(\omega) = \left(r_B\ r'_B(\omega) - r'_B(\omega)\,cos\ \phi_B + r_B\ \phi'_B(\omega)\,sen\ \phi_B\right)^2 \quad (18)$$

Step 5: Choose the frequency and the corresponding parameters assigned to the controller. This selection is not trivial, because, as it will be seen in the examples, the tuning for a specified sensitivity can have a unique or a multiple solution. In the case of a unique solution, the condition of minimum of function FCT is sufficient in order to determine it. However in the case of a multiple solution the search inside a tolerance is the only way to find several zones that contain the different solutions. It is also necessary to use an auxiliary criterion (i.e. the absolute minimum FCT, the maximum integral gain, or the relative minimum FCT in the high frequencies zone) to select the frequency solution.

If the tangency condition of Aström and Hägglund (1995) is used it is only necessary to modify step 4 lightly calculating the magnitude $r(\omega)$ and the phase $\phi(\omega)$ of $G_p(j\omega)$ and their derivatives in the vicinity of the design frequency, by applying the following quadratic function:

$$FCT(\omega) = \left(a(\omega,\theta)\,K_P + b(\omega,\theta)\,K_I + c(\omega,\theta)\,K_D\right)^2 \quad (19)$$

where $a(\omega,\theta)$, $b(\omega,\theta)$ and $c(\omega,\theta)$ are evaluated using the expressions (8).

4 Examples

Let be the third order process evaluated by Persson (1992):

$$G_p(s) = \frac{0.3}{(s+0.3)(s^2+0.6s+1)}$$

The problem consists in tuning a PI controller with the following specifications: M_s = 2 and θ_B = 30º. Figure 6 shows how the tangency condition FCT varies in the whole range of frequencies where there is contact with point B. With a logarithmic sweeping of 100 points, only in the range of 0.53 to 0.57 rd/s does the tangency condition remain below the chosen tolerance (0.001). The minimum of FCT is presented at the frequency ω_c = 0.553 rd/s and the controller parameters K_P = 0.36, T_I = 0.69 and T_D = 0 are obtained.

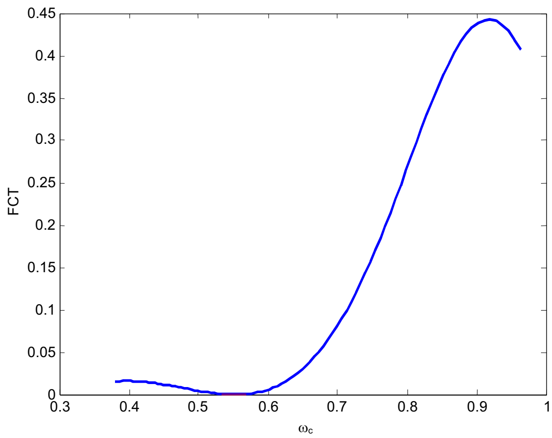

Figure 6: Tangency condition with the PI controller

Figure 7 shows a graph in the parametric plane (K_P,K_I) with all the couples that allow contact with the circle M_s = 2 at the point θ_B = 30º. Those couples that also allow tangential contact with a tolerance less than 0.001 are marked with circles. In the crossing point of the horizontal and vertical lines we find the couple of parameters that is able to minimize the tangency condition.

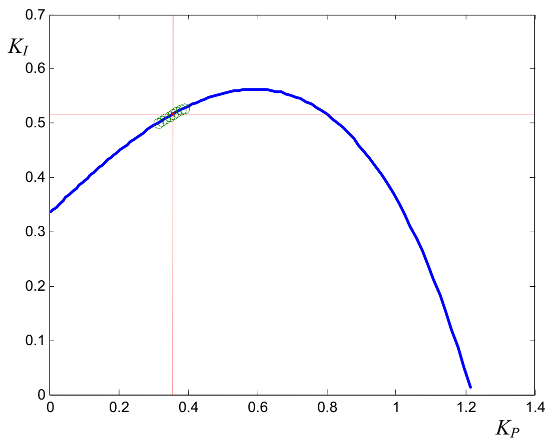

Figure 7: Tuning for maximum sensitivity

Figure 8 shows the Nyquist diagram of the process and the result of the tuning. The circle corresponds to the specification of sensitivity and the radius is associated to point *B* in which the contact and the tangency with this circle take place. A more accurate numeric analysis allows to check that the desired sensitivity specification has been obtained. Figure 9 shows the process output and the control signal responses to steps in the set point and load disturbances.

Figures 10 and 11 show the results for a PID controller with ratio $\alpha = T_D/T_I$ =0.1. Due to the form of the FCT tangency condition, it is observed that now it is possible to contact with point B in a logically bigger range of frequencies, and that two sub ranges exist where the tangency condition remains below the tolerance chosen (0.01); these sub ranges are [0.54 0.58] and [1.18 1.20]. The circles marked in the parametric plane (K_P, K_I) allow to observe where the two families of controllers meet the specifications.

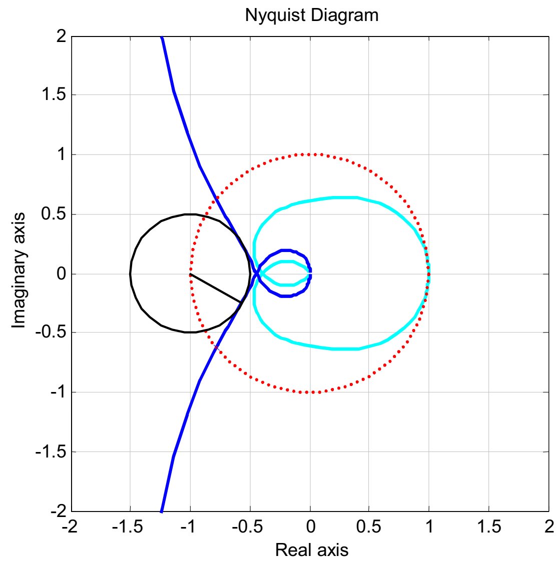

Figure 8: Tuning for maximum sensitivity

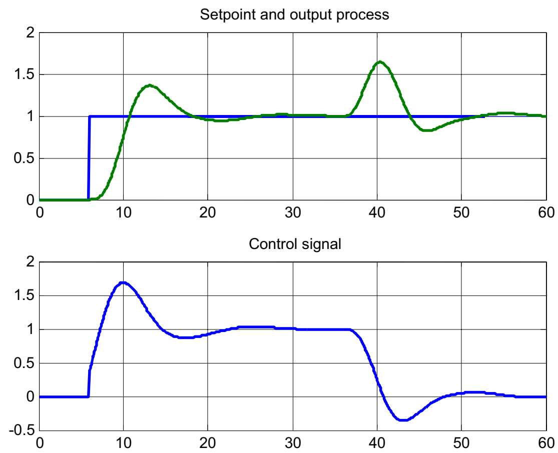

Figure 9: Process output and control signal response

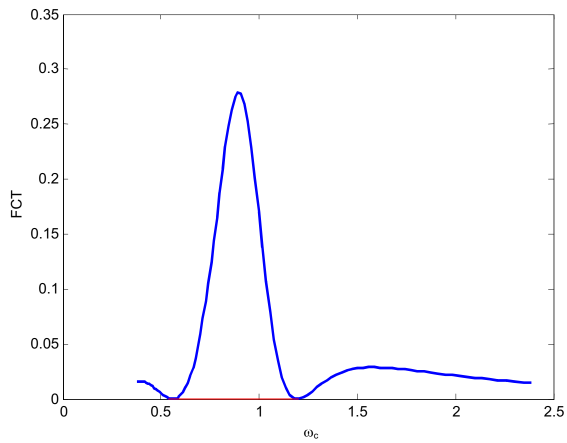

Figure 10: Tangency condition for the PID controller

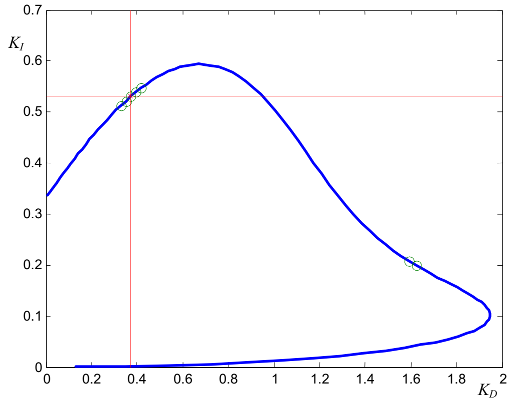

Figure 11: Tangency condition for the PID controller

Figure 12 shows a comparative graph between two possible PID tuning; the first one corresponds to the absolute minimum of FCT at ω_c = 0.561 rd/s, and the second to the relative minimum of the other subrange at ω_c = 1.20 rd/s.

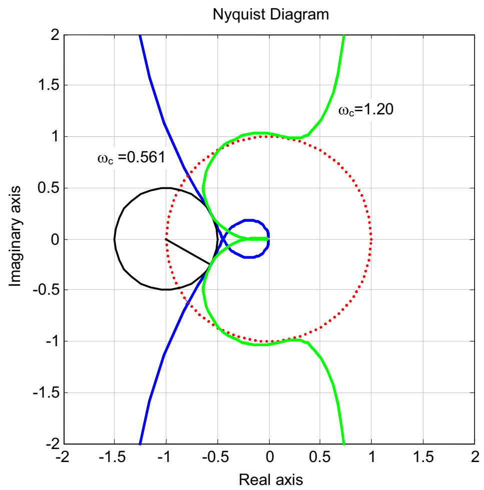

Figure 12: Tangency condition for the PID controller

With the controller parameters of the first tuning, K_P = 0.37, T_I = 0.7 and T_D = 0.07, almost identical to the one obtained with the PI controller, the specified sensitivity is obtained. With the controller parameters of the second tuning, K_P = 1.62, T_I = 8.2 and T_D = 0.82 the specified sensitivity is also obtained. But by observing the comparative graph of the time response in Figure 13 it can be concluded that the first tuning is preferable to the second, because it allows to reach the reference and to reject the load disturbance in a shorter time and with a smaller control effort.

5. Conclusions

The objective of this paper is to present an alternative method to the one proposed by Åström and Hägglund (1995) for the tuning of PID controllers with specification of sensitivity margin M_s. The method is based on the following steps: 1º) to delimit the range of frequencies in which it is possible to contact with a specific point of a circle of radius M_s^{-1} centered at -1, 2º) to explore the frequencies where this contact takes place tangentially, because these frequencies will only be possible solutions and 3º) to choose one of the possible solutions. Some examples have been selected in order to show the main features of the proposed method.

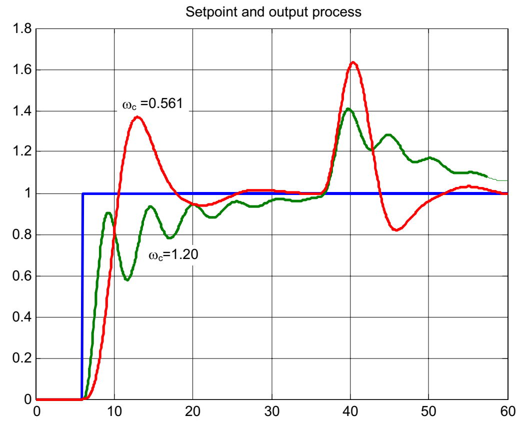

Figure 13: Set point and load disturbances responses

Acknowledgements

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