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COMPARISON BETWEEN THE EXTREME POINT RESULTS TO STABILIZE AN INTERVAL PLANT

R. Hernández, R. Dormido, F. Morilla

Dept. Informática y Automática, UNED
C/ Senda del Rey s/n, 28040 Madrid, Spain
e-mail: rhdez@dia.uned.es

Abstract: Nowadays there are several extreme point results to solve the design problem for an interval plant in closed loop. The aim of this paper is to compare the conservadurism between these methods. First a new set of virtual polynomials for this problem based on Bialas and Garloff polynomials to stabilize an edge is proposed. Then its conservadurism is studied with regard to another conservative method based on 32 virtual polynomials, concluding that it is a less conservative method but the computational cost required is higher. With the purpose of showing a global perspective of the differents methods they are compared with the Kharitonov polynomials of the smallest interval polynomial that contains the characteristic polynomial polytope, and with the 32 CB segments, concluding that the set formed by the 32 virtual polynomials is a good agreement between conservadurism and computational cost. *Copyright © 1999 IFAC*

Keywords: Robust Control, Controllers, Extreme Point Results.

1. INTRODUCTION

Robust control based on extreme point results has been the object of special interest throughout the last decade. Since the introduction in western literature of Kharitonov's Theorem by R. Barmish in 1983 very interesting results have appeared specially on the analysis problem. From the synthesis point of view there are a few extreme point results (Barmish, 1993) for the specially interesting problem called interval plant paradigm.

The most significant early results are Ghosh's works (1985) for the pure gain compensator, Hollot and Yang (1990) and Barmish's 16 plants Theorem (Barmish, *et al.*, 1992) for first order controllers. For controllers with an order greater than one the first method consists of building the four Kharitonov polynomials of the smallest interval polynomial that contains the polytope formed by the closed loop characteristic polynomials. Ghosh formulated these polynomials for positive controllers (Ghosh, 1985), and they were generalized later (Hernández, *et al.*1996).

The first chance to stabilize an interval plant using controllers of any order is to stabilize the 32 CB segments (Chapellat and Battacharyya, 1989). This is a no conservative result which allows to find all the controllers that stabilize the family. Nevertheless, other results have been developed in order to reduce the computational cost. These results make use of virtual polynomials, so called because they do not belong to the family. So, Djaferis shows that it is sufficient to assure the stability of 64 virtual polynomials (Djaferis, 1993). By developing an alternative construction it is shown that only 32 virtual polynomials are sufficient to guarantee the stability of the whole family, and

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this number can be reduced depending on the argument of the numerator and denominator polynomials of the controller (Hernández and Dormido S., 1995). Not only the generalized Ghosh's polynomials but also the 32 virtual polynomials are conservative results because they do not allow to find all the controllers that stabilize the family but some of them. As it has been shown the advantage of the 32 virtual polynomials, except for the cases in which both methods coincide, is that they are less conservative than the Kharitonov polynomials of the smallest interval polynomial that contains the characteristic polynomial polytope (see (Hernández, *et al.*, 1996 for details). Obviously its disadvantage is the higher number of polynomials to be stabilized.

These two methods are not the only results to stabilize an interval plant using a set of virtual polynomials. In fact, if we bear in mind one of the first extreme point results developed by Bialas and Garloff (1985) to stabilize an edge, a set of 80 virtual polynomials which guarantee the stability of the interval plant can be obtained.

The aim of this paper is not only to compare the three sets of virtual polynomials, searching theirs properties, but also to establish the conservatism of each method with regard to the others.

2. PREVIOUS

Let $P(s,a,b)$ be a proper plant of real coefficients of the form

$$P(s,a,b) = N_p(s,a)/D_p(s,b) \quad (1)$$

where $N_p(s,a)$ and $D_p(s,b)$ are interval polynomials

$$N_p(s,a) = a_m s^m + ... + a_0; \quad D_p(s,b) = b_n s^n + ... + b_0 \quad (2)$$

$$a \in A = \left\{a : a_i^- \le a_i \le a_i^+, i = 0,...,m\right\}$$

$$b \in B = \left\{b : b_i^- \le b_i \le b_i^+, i = 0,...,n\right\}$$

with $m \ge 1$, $m \le n$, and where $a = [a_0, a_1, ..., a_m]$, $a_m \ne 0$, and $b = [b_0, b_1, ..., b_n]$, $b_n \ne 0$, are the uncertainty parameters.

Let the controller $C(s) = N_c(s)/D_c(s)$. The family of characteristic polynomials associated to the closed loop system configuration formed by $P(s,a,b)$ and $C(s)$ with unity feedback is

$$\delta(s) = N_c(s)N_p(s,a) + D_c(s)D_p(s,b) \quad (3)$$

The stabilization problem consists of proving that the polytope of polynomials (3) is Hurwitz.

Similarly the Kharitonov polynomials of the interval polynomial will be expressed in terms of their even and odd parts

Family $N_p(s)$: (4)

$$k_{n1}(s) = p_{emin}(s) + p_{omin}(s), \quad k_{n2}(s) = p_{emax}(s) + p_{omin}(s)$$

$$k_{n3}(s) = p_{emax}(s) + p_{omax}(s), \quad k_{n4}(s) = p_{emin}(s) + p_{omax}(s)$$

where $p_{emin}(s) = a_0^- + a_2^+ s^2 + ...$, $p_{emax}(s) = a_0^+ + a_2^- s^2 + ...$

$$p_{omin}(s) = a_1^- s + a_3^+ s^3 + ..., \quad p_{omax}(s) = a_1^+ s + a_3^- s^3 + ...$$

Family $D_p(s)$: (5)

$$k_{d1}(s) = q_{emin}(s) + q_{omin}(s), \quad k_{d2}(s) = q_{emax}(s) + q_{omin}(s)$$

$$k_{d3}(s) = q_{emax}(s) + q_{omax}(s), \quad k_{d4}(s) = q_{emin}(s) + q_{omax}(s)$$

where $q_{emin}(s) = b_0^- + b_2^+ s^2 + ...$, $q_{emax}(s) = b_0^+ + b_2^- s^2 + ...$

$$q_{omin}(s) = b_1^- s + b_3^+ s^3 + ..., \quad q_{omax}(s) = b_1^+ s + b_3^- s^3 + ...$$

The conservative result reported by Bialas and Garloff to stabilize an edge is the following:

Theorem 1: (Bialas and Garloff,1985) Let $f_0(s)$ and $f_1(s)$ be two polynomials of real coefficients with degree n. The edge with vertexes in $f_0(s)$ and $f_1(s)$ is stable if and only if the four polynomials below are stable.

$$f_0(s);\ f_1(s);$$

$$g_1(s) = Even[f_0(s)] + Odd[f_1(s)];$$

$$g_2(s) = Even[f_1(s)] + Odd[f_0(s)]$$

3.CONSTRUCTION OF THE VIRTUAL POLYNOMIALS

Theorem 1 establishes that stabilization of an edge is equivalent to stabilize 4 polynomials. Two of them are its vertexes polynomials and the other couple are virtual polynomials.

Attending to the Box Theorem (Chapellat and Battacharyya, 1989), $\delta(s)$ is stable if and only if the 32 CB segments are stables. To apply the Bialas and Garloff theorem to each CB segment it will be necessary not only two virtual polynomials for each edge but also their vertexes polynomials. So we will have 64 virtual polynomials plus 16 polynomials due to the 16 Kharitonov plants.

In order to apply Bialas and Garloff theorem, we need to know the vertexes of the edges to stabilize. With the purpose of calculating them the following notation is presented. Let $p(s)$ a given polynomial. The $Even[\,]$ and

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$Odd[]$ functions are defined as those that return the even and odd parts of $p(s)$, respectively. If the polynomial is evaluated at $s = j\omega$, the $Even[]$ function is a real number and $Odd[]$ is a complex number without the real part.

The even and odd parts of $N_c(s)$ ($D_c(s)$) will be denoted by $N_{ce}(s)$ ($D_{ce}(s)$) and $N_{co}(s)$ ($D_{co}(s)$) respectively

$$N_c(s) = N_{ce}(s) + N_{co}(s); \quad D_c(s) = D_{ce}(s) + D_{co}(s);$$

The value set of $\delta(s)$, $\Delta(\omega)$, is a polygon in the complex plane formed by the direct sum of the two parpolygons $\Delta(\omega) = \Delta_1(\omega) + \Delta_2(\omega)$ where

$$\Delta_1(\omega) = \{N_c(j\omega)N_p(j\omega, a)\} \; \Delta_2(\omega) = \{D_c(j\omega)D_p(j\omega, b)\} \quad (6)$$

$\Delta_1(\omega)$ ($\Delta_2(\omega)$) is a paralelogram with vertexes $p_i(s) = N_c(s)k_{ni}(s)$ ($q_i(s) = D_c(s)k_{di}(s)$) where $k_{ni}(s)$ ($k_{di}(s)$), $i = 1,2,3,4$ are the Kharitonov's polynomials of $N_p(s)$ ($D_p(s)$) and whose sides have the slopes

$$\alpha_1 = \frac{\operatorname{sen}\theta}{\cos\theta} \; and \; \alpha_2 = \frac{-\cos\theta}{\operatorname{sen}\theta}$$

where $\theta = \theta(\omega)$ is the argument of the polynomial $N_c(s)$ ($D_c(s)$) evaluated at $s = j\omega$ for a given ω (see (Hernández and Dormido S., 1995 for details). See Fig 1.

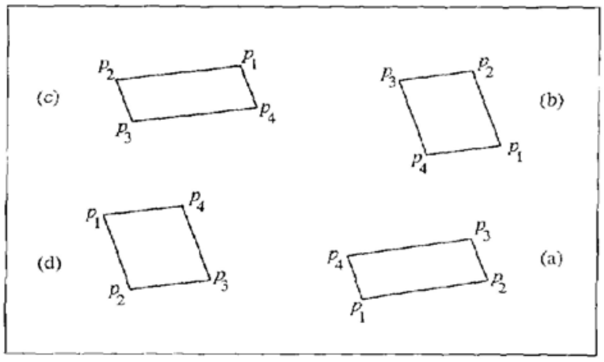

Fig 1. Different positions of $\Delta_1(\omega)$ (Similarly for $\Delta_2(\omega)$ replacing p_i by q_i, $i = 1,2,3,4$)

When each paralelogram $\Delta_1(\omega)$ and $\Delta_2(\omega)$ can be found at any of the four positions showed at Fig 2, the edges which oberbound the image $\Delta(\omega)$ are the 32 CB segments. Virtual polynomials' construction is immediate. For instance, the two virtual polynomials $bg_{11}(s)$ and $bg_{12}(s)$ are those which correpond to the segment whose vertexes are (see Fig 2)

$$v_0 = N_c k_{n1} + D_c k_{d1} \text{ and } v_1 = N_c k_{n1} + D_c k_{d2} \quad (7)$$

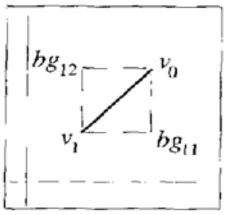

Fig 2. Overbounding of the Bialas and Garloff polynomials

Developing the real and imaginary parts and grouping terms (removing the dependence in s to simplify the notation) the following is obtained

$$bg_{11} = Even[v_0] + Odd[v_1] = N_c k_{n1} + D_{ce}k_{de1} + D_{co}k_{do1} +$$
$$+ D_{ce}k_{do2} + D_{co}k_{de2} = N_c k_{n1} + D_{ce}(k_{de1} + k_{do2}) + D_{co}(k_{do1} + k_{de2})$$

By the expresions (5) $k_{do2} = k_{do1} = q_{omin}$. Then

$$bg_{11} = N_c k_{n1} + D_{ce}(k_{de1} + k_{do1}) + D_{co}(k_{do2} + k_{de2})$$
$$= N_c k_{n1} + D_{ce}k_{d1} + D_{co}k_{d2}$$

Similarly

$$bg_{12} = Even[v_1] + Odd[v_0] = N_c k_{n1} + D_{ce}k_{d2} + D_{co}k_{d1}$$

In the same way the rest of the Bialas and Garloff's virtual polynomials for each CB segment can be calculated and the following 64 polynomials presented in compact notation are obtained (8)

$$bg_{i1}^C = Ck_{ni} + D_{pe}k_{d1} + D_{po}k_{d2}, \; bg_{i2}^C = Ck_{ni} + D_{ce}k_{d2} + D_{co}k_{d1}$$
$$bg_{i3}^C = Ck_{ni} + D_{pe}k_{d2} + D_{po}k_{d3}, \; bg_{i4}^C = Ck_{ni} + D_{pe}k_{d3} + D_{po}k_{d2}$$
$$bg_{i5}^C = Ck_{ni} + D_{pe}k_{d3} + D_{po}k_{d4}, \; bg_{i6}^C = Ck_{ni} + D_{pe}k_{d4} + D_{po}k_{d3}$$
$$bg_{i7}^C = Ck_{ni} + D_{pe}k_{d4} + D_{po}k_{d1}, \; bg_{i8}^C = Ck_{ni} + D_{pe}k_{d1} + D_{po}k_{d4}$$

If $C = N_c$ and i=1,2,3,4 32 virtual polynomials are obtained and when $C = D_c$ the others 32 are obtained.

The following result can be established:

Theorem 2: If the 16 characteristic polynomials of the 16 Kharitonov plants and the 64 Bialas and Garloff's virtual polynomials are stable then the closed loop system formed by an interval plant and a controler is stable. ◊

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Proof: It is obvious: The stability of the CB segments is a necessary and sufficient condition to guarantee the stability of the closed loop system. By Theorem 1 the stability of the 80 characteristic polynomials is a sufficient condition to stabilize the edges, hence if they are stable all the family is stable.

4. COMPARISON

Due to the stability of the 80 virtual polynomials in Theorem 2 is a sufficient but not necessary condition to guarantee the stability of the closed loop family, this is a conservative result. The following theorem points out this conservatism.

Theorem 3: Let SC_{32}, SC_{bg}, SC be the set of controllers which stabilize the family (3) through the 32 virtual polynomials (Hernández and Dormido, 1995), the 80 Bialas and Garloff polynomials and the CB segments respectively. Then

$$SC_{32} \subset SC_{bg} \subset SC \qquad \Diamond$$

This theorem can be proved in a simple way from the behaviour of the value set, taking into account the following Lemmas.

Lemma 1. (See (Hernández, *et al.*, 1996) for prove). The controllers $C(s)$ designed stabilizing the 32 virtual polynomials, are those so that the value set of the family of closed loop characteristic polynomials $\delta(s)$ is applied, for each ω in two quadrants as maximum. ◊

Lemma 2. The controllers $C(s)$ designed stabilizing the 80 polynomials SC_{bg} are those so that the CB segments are applied, for each ω in two cuadrants as maximum. ◊

Proof. (Similar to the one in Lemma 1) It is assumed that $\delta(s)$ is stable. Thus it satisfies the argument principle. The overbounding in the complex plane consists in including each segment in a rectangle with stable vertexes for each ω (Fig 2). If any CB segment can be placed in three quadrants at some ω, then at least one vertex does not satisfy the argument principle and is unstable. Fig3.a shows this behaviour. However, if each CB segment does not enter into a quadrant until it is enterely in the previous one, then the Bialas and Garloff polynomials are stable (Fig 3.b).

Proof. It is obvious from the previous lemmas since the set of controllers SC_{32} will not include those SC_{bg} controllers which apply the value set in three quadrants for some ω (with CB segments in two quadrants as maximum), as for example the one which appears in the Fig 3.b. Then $SC_{32} \subset SC_{bg}$. The matter that $SC_{bg} \subset SC$ is proved in Theorem 1 of Bialas and Garloff ◊

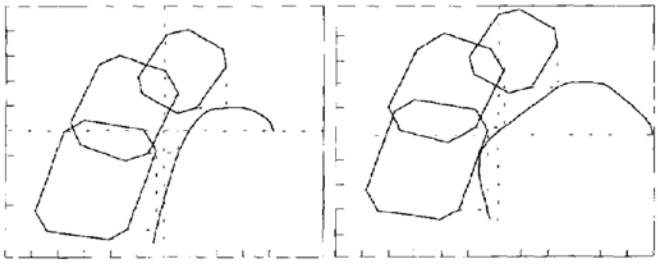

Fig 3. (a) Family stable and Bialas and Garloff virtual polynomial unstable. (b) Family stable and Bialas and Garloff virtual polynomial stable ◊

Proof. It is obvious from the previous lemmas since the set of controllers SC_{32} will not include those SC_{bg} controllers which apply the value set in three quadrants for some ω (with CB segments in two quadrants as maximum), as for example the one which appears in the Fig 3.b. Then $SC_{32} \subset SC_{bg}$. The matter that $SC_{bg} \subset SC$ is proved in Theorem 1 of Bialas and Garloff ◊

The method based on Bialas and Garloff virtual polynomials is less conservative, in the sense that more controllers can be obtained, than the one based on the 32 virtual polynomials, but the computational cost required is higher because more polynomials are required to be stable.

Taking into account Theorem 3 and Theorem 2 in (Hernández, *et al.*, 1996) the following Corollary can be established.

Corollary. Let SC_4, SC_{32}, SC_{bg}, SC be the set of controllers which stabilize the family (3) using the four Kharitonov's polynomials of the smallest interval polynomial that contains the characteristic polynomial polytope in closed loop, the 32 virtual polynomials (Hernández and Dormido S., 1995), the 80 Bialas and Garloff polynomials and the CB segments respectively. Then

$$SC_4 \subset SC_{32} \subset SC_{bg} \subset SC \qquad \Diamond$$

In the comparative study between SC_4 y SC_{32} (Hernández, *et al.*, 1996), the conditions which have to be satisfied so that both sets coincide and the controllers' structures which guarantee such coincidence were presented. It is also possible that the sets SC_{32} and SC_{bg} coincide, such as explain the following lemma.

Lemma 3. $SC_{32} = SC_{bg}$ if and only if $N_c = s^l$ o $D_c = s^l$, l=integer. ◊

Proof. The 32 virtual polynomials' method is based on overbounding the value set with the minimal rectangle

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which include it for each ω. If $N_c = s^l$ ($D_c = s^l$) then taking into account the expressions (6) the value set $\Delta_1(\omega)$ ($\Delta_2(\omega)$) is a rectangle for each ω. In this case the value set formed by $\Delta_1(\omega)+\Delta_2(\omega)$ will have the edges parallel to the axes (fig 4.a), so that there are not Bialas and Garloff virtual polynomials to this edges and the Bialas and Garloff virtual polynomials which coincide with the virtual polynomials of SC_{32} are sufficient to stabilize the family (the rest are applied inside the value set so they are superfluous).

If $N_c \neq s^l$ and $D_c \neq s^l$ then both sets of polynomials do not coincide (fig 4.b).

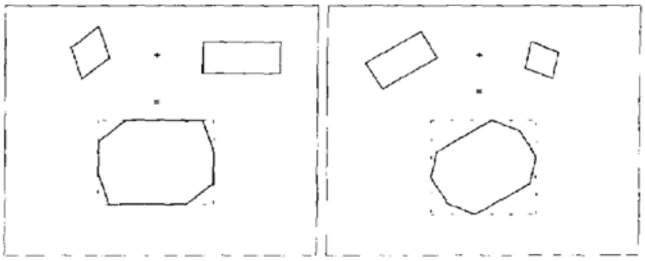

Fig 4. (a) Coincidence (b) Not coincidence between the 32 virtual polynomials and the Bialas and Garloff virtual polynomials ◊

Table 1 summarizes the conservadurism properties of the controllers designed with each method.

Table 1. Properties of the controllers designed with each method (presented with decresing conservatism)

Method	Controllers found
Kharitonov polynomials of the smallest interval polynomial that contains the characteristic polynomial polytope	Some of the controllers such that $\Delta(\omega)$ is applied in two quadrants as maximum for each ω
32 virtual polynomials	All the controllers such that $\Delta(\omega)$ is applied in two quadrants as maximum for each ω
80 Bialas and Garloff polynomials	All the controllers such that the CB segments are applied in two quadrants as maximum for each ω
No conservative	All the controllers which stabilize the closed loop system

As it is shown, any of the conservative methods let find robust controllers so that a CB segment is applied in three quadrants for any ω. For instance, a controller with the value set $\Delta(\omega)$ showed in Fig 5 can just be designed with a no conservative method.

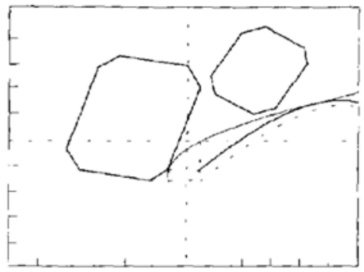

Fig 5. Stable family and 32 virtual polynomials (- - -) and Bialas and Garloff virtual polynomials (——) unstable.

5. EXAMPLES

In this section, two examples which illustrate the resuls obtained in this note are presented.

Example 1 (Hernández, et al., 1998). Consider the interval plant $P(s,a,b)$

$$P(s,a,b) = \frac{a_1 s + a_0}{s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0} \qquad (9)$$

where $20 \le a_1 \le 60$, $120 \le a_0 \le 128$, $10 \le b_3 \le 20$, $45 \le b_2 \le 60$, $30 \le b_1 \le 50$ y $-10 \le b_0 \le 10$.

Clearly, this interval plant is unstable. To stabilize the family we have a closed loop system, with a second degree controller, $C(s)$, described by $C(s) = \frac{s+1}{s^2 + d_1 s + d_0}$ where d_1 and d_0 are the design's parameters. For this kind of controllers there is no, in general, extreme point results in terms of the 16 Kharitonov's plants.

Fig 6. shows the controllers which are obtained when the closed loop polynomial polytope is stabilized with the different methods analyzed in the paper (they have been calculated after Ackermann's diagram in the space parameter (Ackermann, 1980)).

As can be seen the set SC_{bg} in not much bigger than SC_{32}, however SC is remarkably bigger than any of the sets calculated with the conservative methods, obviously the computational cost requiered is much bigger. In conclusion, it can be seen that using 32 virtual polynomials a good agreement between computational cost and conservadurism is obtained.

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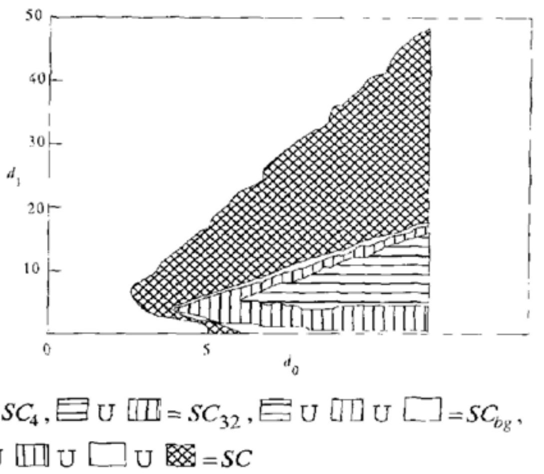

Fig 6. Controllers which stabilize the family with conservative methods

Example 2. Let again the interval plant (9) and the PI controller

$$C(s) = K_1 + \frac{K_2}{s},\ K_1 > 0,\ K_2 > 0$$

First we should bear in mind that for first order controllers the 16 plants theorem (Barmish, *et al.*, 1992) can be applied and $SC_4 = SC_{32}$ (Hernández, *et al.*, 1998).

Like $K_1 > 0$, $K_2 > 0$ then $C(s) = \dfrac{K_1 s + K_2}{s}$ has argument of the numerator polynomial between 0 y $\pi/2$ and the argument of the denominator polynomial is always $\pi/2$. Because of the constant argument of the denominator the image $\Delta_2(\omega)$ is a Kharitonov's rectangle $\forall\omega$. Then, the Bialas and Galoff's virtual polynomials which are not superfluous coincide with the sufficients of the 32 virtual polynomials.

In conclusion, *with PI controllers, the three conservative methods coincide,* so there is no advantage between them.

6. CONCLUSION

This paper has presented the virtual polynomials which stabilize the system formed by a controller and an interval plant in closed loop based on Bialas and Garloff virtual polynomials to stabilize an edge and the CB segments. They have been presented so that the comparison with other known results can be done. So it has been proved that controllers which stabilize Bialas and Garloff polynomials are those such that the CB segments are applied in two quadrants as maximum for each ω.

Then its conservadurism has been studied with regard to the 32 virtual polynomials, concluding that it is a less conservative method, that is, $SC_{32} \subset SC_{bg}$, and taking in account the other known methods it is proved that $SC_4 \subset SC_{32} \subset SC_{bg} \subset SC$.

It is also proved that to use Bialas and Garloff polynomials is less conservative and the computational cost required is appreciabily bigger. Even so it can be established that 32 virtual polynomials represent the best agreement between computational cost and conservadurism.

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