THE SAMPLING PERIOD AS A CONTROL PARAMETER

I. LOPEZ, S. DORMIDO and F. MORILLA

UNED, Dpto de Informática y Automática, Avda Senda del Rey s/n, 28040 Madrid, Spain

Abstract. When a computer is used in the control of continuous processes, there arises an important parameter, sampling period T. In the analysis and design of digital control systems it is usual to transfer the results obtained for analog systems, doing T small. This procedure, valid for many processes, in some others can cause control deterioration and, besides, the advantages inherent to computer controlled systems can be lost. These are the purposes of the present paper: 1.-Give some rules for the selection of T as a control parameter. 2.-Compare the result obtained for the continuous case.

Key Words. PID control; Sampled data systems; Control system design; Digital control; Time-domain analysis.

1. INTRODUCTION

When a computer is used in the control of continuous processes, an important parameter is the sampling period T, because it has a great influence on the time response characteristics of the system (Åström and Wittenmark 1990).

The problem which arises is, once it is known the model of the process Gm(s), the time response characteristics and the controller Gc(z), to obtain the tuning formulas which allow to adjust the parameters of Gc(z) and the sampling period.

For the adjustment of the control parameters it can follow two procedures:

1.-To design the controller as if it were continuous and to adjust T, by means of simulation, in such a manner that the system response will be similar to the response obtained by continuous control.

2.-To consider the discrete behaviour of the system and to adjust T as a control parameter (Dormido *et al.* 1991).

The purpose of the present paper is twofold: 1.-To obtain the tuning formulas by using the tuning method by continuous and discrete pole placement (Åström and Hägglund 1988). 2.-To compare the results obtained, when you use both, in computer control.

The paper is organized as follows: The controllers and models to use are introduced in sections 2 and 3. The tuning method, formulas and criteria, are showed in sections 4, 5 and 6. Finally in sections 7 and 8 some examples and conclusions are given.

2. CONTROLLER

Due to its wide in industrial use, the controller will be of a PID type, in some of its versions P, PI, PD or PID (Shinskey 1989).

The basic algorithm of PID continuous control is

$$U(s) = Kc\left(1 + \frac{1}{sTi} + sTd\right)E(s) \tag{1}$$

and its discrete version can be expressed as

$$U(z) = Kc\left(1 + \frac{T}{Ti}\frac{z}{z-1} + \frac{Td}{T}\frac{z-1}{z}\right)E(z) \tag{2}$$

The continuous and discrete controllers can be given by the following expression

$$Gc(s) = S(s) / R(s) \tag{3}$$

$$Gc(z) = S(z) / R(z) \tag{4}$$

where S(s), R(s) and S(z), R(z) are given in tables 1 and 2 respectively.

Table 1 Continuous Controllers

Gc(s)	S(s)	R(s)
P	$S(s) = s_0$ $s_0 = Kc$	1
PI	$S(s) = s_0 s + s_1$ $s_0 = Kc \quad s_1 = \frac{Kc}{Ti}$	s
PD	$S(s) = s_0 s + s_1$ $s_0 = KcTd \quad s_1 = Kc$	1
PID	$S(s) = s_0 s^2 + s_1 s + s_2$ $s_0 = KcTd \; s_1 = Kc \; s_2 = \frac{Kc}{Ti}$	s

Table 2 Discrete Controllers

Gc(z)	S(z)	R(z)
P	$S(z) = s_0$ $s_0 = Kc$	1
PI	$S(s) = s_0 z + s_1$ $s_0 = \frac{Kc(Ti + T)}{Ti} \quad s_1 = -Kc$	z-1
PD	$S(s) = s_0 z + s_1$ $s_0 = \frac{Kc(T + Td)}{T} \quad s_1 = -\frac{KcTd}{T}$	z
PID	$S(s) = s_0 z^2 + s_1 z + s_2$ $s_0 = \frac{Kc(T\,Ti + Td\,Ti + T^2)}{Ti\,T}$ $s_1 = \frac{Kc(T + 2Td)}{T} \quad s_2 = \frac{KcTd}{T}$	z^2-z

3. PROCESS MODELS

The processes will be continuous and described by means of

$$Gm(s) = B(s) / A(s) \tag{5}$$

If the behaviour of the D/A converter is that of a zero order holder, $Go(s)=(1-e^{-Ts})/s$, the discrete models will be obtained from (Phillips and Nagle 1984)

$$Gm(z) = (1 - z^{-1}) Z[Gm(s) / s] \tag{6}$$

resulting

$$Gm(z) = B(z) / A(z) \tag{7}$$

In the following the models of first and second order will be considered:

1.-Model 1

$$A(s) = s + a_1,\ B(s) = b_1 \tag{8}$$

$$A(z) = z + a_1',\ B(z) = b_1' \tag{9}$$

2.-Model 2

$$A(s) = s^2 + a_1 s + a_2,\ B(s) = b_1 \tag{10}$$

$$A(z) = z^2 + a_1' z + a_2',\ B(z) = b_1' z + b_2' \tag{11}$$

4. TUNING METHODS

Given Gm(s), Gc(z) and the time response characteristics, the parameters of Gc(z) and the sampling period T will be obtained by means of the following methods.

4.1. *Method by continuous pole placement*

Step 1: Calculate Gc(s) from Gc(z).

Step 2: From Gl(s) = Ga(s) / (1+Ga(s)) with Ga(s)=Gc(s)Gm(s), to deduce the system response Y(t). The Gl(s) parameters according to the characteristics specified can be obtained from y(t), and so the roots of the characteristic eq.

Step 3: Equalizing coefficients, to obtain from 1+Ga(s)=0 the relations between the roots obtained in step 2 and the parameters of Gm(s) and Gc(s).

Step 4: Calculate Gc(z) parameters and to adjust T in such a manner that the characteristics specified are verified.

4.2. *Method by discrete pole placement*

Step 1: Calculate Gm(z) from Gm(s) and (7).

Step 2: .-The roots location will be designed by the specified characteristics if these can be determined in the discrete domain, or, if the characteristics are referred to the system continuous behaviour from the roots location of a continuous system Gle(s), equivalent to Gl(z). In this case the sampling period T permit to specify one additional characteristic or the number of samples taken between different time instants.

Step 3: -Similarly as in the continuous case, to obtain the relations between roots location and the parameters of Gm(z) and Gc(z), and calculate Gc(z) parameters and T (Roffel *et al.* 1989).

5. TUNING FORMULAS

Stable systems with monotonous responses (if the order of the system is smaller that three) and/or underdamped response (if the order of the system is grater than one), are formed by controllers type PID and processes given by the models 1 and 2 given in section 3. The systems to be considered will be model 1 with P and PI controllers, and model 2 with P, PI, PD and PID controllers.

Table 3 Characteristic equations

SYST	CHARACTERISTIC EQ.
1 P	$s+a_1+s_0b_1=0$ $s+r_1=0$
1 PI	$s^2+(a_1+s_0b_1)s+s_1b_1=0$ $(s+r_1)(s+r_2)=0$ or $s^2+p_1s+p_2=0$
2 P	$s^2+a_1s+a_2+s_0b_1=0$ $(s+r_1)(s+r_2)=0$ or $s^2+p_1s+p_2=0$
2 PI	$s^3+a_1s^2+(a_2+s_0b_1)s+s_1b_1=0$ $(s+r_1)(s^2+p_1s+p_2)=0$
2 PD	$s^2+(a_1+s_0b_1)s+a_2+s_1b_1=0$ $s^2+p_1s+p_2=0$
2 PID	$s^3+(a_1+s_0b_1)s^2+(a_2+s_1b_1)s+b_1s_2=0$ $(s+r_1)(s^2+p_1s+p_2)=0$

5.1. *Continuous Control*

From (3) and (5) the characteristic eq. of the continuous system can be expressed

$$R(s)A(s)+S(s)B(s)=0 \qquad (12)$$

R(s), S(s) are shown in table 1 and A(s), B(s) in (8) and (10). In table 3 the characteristic eq. for different systems are given, according to the parameters of Gm(s) and Gc(s). The root placements are obtained from the characteristics specified. As the system response will be stable and monotonous if the roots of the characteristic eq. are real, and underdamped if there are complex roots, it will be

$$r_i=1/\tau_i,\ p_1=2\delta\omega_n,\ p_2=\omega_n^2 \qquad (13)$$

being τ_i the time constant, $0<\delta<1$ the damping ratio and ω_n the natural frequency.

By identifying coefficients in table 3 it is obtained table 4, where the s_i controller coefficients are related to the parameters of the models a_i, b_i and the roots placement r_i, p_i.

Table 4 Controller coefficients

SYST	S_0	S_1	S_2
1 P	$\frac{r_1-a_1}{b_1}$	-	-
1 PI	$\frac{r_1+r_2-a_1}{b_1}$ or $\frac{p_1-a_1}{b_1}$	$\frac{r_1r_2}{b_1}$ or $\frac{p_2}{b_1}$	-
2 P	$\frac{r_1r_2-a_2}{b_1}$ or $\frac{p_2-a_2}{b_1}$	-	-
2 PI	$\frac{r_1p_1+p_2-a_2}{b_1}$	$\frac{r_1p_2}{b_1}$	-
2 PD	$\frac{p_1-a_1}{b_1}$	$\frac{p_2-a_2}{b_1}$	-
2 PID	$\frac{r_1+p_1-a_1}{b_1}$	$\frac{p_2+r_1p_1-a_2}{b_1}$	$\frac{r_1p_2}{b_1}$

Table 5 Tuning formulas

SYST	SPECIFICATIONS	TUNING F.
1 P	r_1	$Kc=s_0$
1 PI	r_1, r_2 or p_1, p_2	$Kc=s_0$ $Ti=\frac{s_0}{s_1}$
2 P	one of r_1, r_2 or p_1, p_2 $a_1=r_1+r_2$ or $a_1=p_1$	$Kc=s_0$
2 PI	two of r_1, p_1, p_2 $a_1=r_1+p_1$	$Kc=s_0$ $Ti=\frac{s_0}{s_1}$
2 PD	p_1, p_2	$Kc=s_0$ $Td=\frac{s_0}{s_1}$
2 PID	r_1, p_1, p_2	$Kc=s_1$ $Ti=\frac{s_1}{s_2}$ $Td=\frac{s_0}{s_1}$

The parameters of Gc(s) are obtained from tables 1 and 4. In table 5 for different systems are showed the specifications to consider and the tuning formulas. The sampling period T will be obtained, by simulation, in such a manner that the system response will be simalar to the response obtained with a continuous controller.

5.2. *Discrete control*

From (4) and (7) the characteristic eq. of the discrete system is

$$R(z)A(z) + S(z)B(z) = 0 \qquad (14)$$

R(z), S(z) are shown in table 2 and A(z), B(z) in (9) and (11). Analogous by as in the continuous case it is deduced table 6 where the controller coefficients s_i are related to the parameters of the model a_i', b_i' and the roots placement r_i, p_i.

Table 6 Controller coefficients

SYST	S_0	S_1	S_2
1 P	$\frac{r_1 - a_1'}{b_1'}$	-	-
1 PI	$\frac{r_1 + r_2 + 1 - a_1'}{b_1'}$ or $\frac{p_1 + 1 - a_1'}{b_1'}$	$\frac{r_1 r_2 + a_1'}{b_1'}$ or $\frac{p_2 + a_1'}{b_1'}$	-
2 P	$\frac{r_1 r_2 - a_2'}{b_2'}$ or $\frac{p_2 - a_2'}{b_2'}$	-	-
2 PI	$\frac{p_1 + r_1 - a_1' + 1}{b_1'}$	$\frac{p_2 r_1 + a_2'}{b_2'}$	-
2 PD	$\frac{p_1 + r_1 - a_1'}{b_1'}$	$\frac{p_2 r_1}{b_2'}$	-
2 PID	$\frac{p_1 + r_1 + r_2 - a_1' + 1}{b_1'}$	$((r_1 + r_2)p_2 b_2' + r_1 r_2 (p_1 b_2' - p_2 b_1') + a_2 b_2') / b_2'^2$	$\frac{r_1 r_2 p_2}{b_2'}$

The parameters of Gc(z) are obtained from tables 2 and 6. In table 7 are showed for different systems the specifications to consider and the tuning formulas .

If the characteristics of the time response are given directly in the z-plane, they will be considered as r_i and/or p_i specifications. By contrary if the characteristics are referred to the continuous behaviour, the placement roots of the Gl(z) characteristic eq. will be obtained from the roots placement of the characteristic eq. of the continuous system Gle(s). So, when the Gl(z) characteristic eq. present real roots, $z = -r_i$, it is considered that the contribution to the time response system is equivalent to simple poles $s = -1/\tau_i$, in Gle(s). Therefore

$$r_i = -e^{-T/\tau_i} \qquad (15)$$

if Gl(z) characteristic eq. $z^2 + p_1 z + p_2 = 0$, presents complex roots it is considered that the contribution of these to the system response is equivalent to have complex poles $s = -2\delta\omega_n \pm \omega_n\sqrt{1-\delta^2}\,j$ in Gle(s), resulting

$$p_1 = -2e^{-\delta\omega_n T}\cos\left(\omega_n T\sqrt{1-\delta^2}\right),\ p_2 = e^{-2\delta\omega_n T} \qquad (16)$$

In discrete control the samplig period T is fixed by the specifications when the roots number of the characteristic eq. is greater than the number of the control parameters; this will happen for model 2. If the number of roots of the characteristic eq. is the same as the control parameters, T will not be fixed for the specifications; this will happen for model 1, where it will be considered as one more specification.

Table 7 Tuning Formulas

SYST	SP.	TUNING F.
1 P	r_1, T	$Kc = s_0$
1 PI	r_1, r_2, T or p_1, p_2, T	$Kc = -s_1 \quad Ti = -\frac{s_1 T}{s_0 + s_1}$
2 P	r_1, r_2 or p_1, p_2	$Kc = s_0$ $T \rightarrow b_2'(r_1 + r_2 - a_1') - b_1'(r_1 r_2 - a_2') = 0$ $T \rightarrow b_2'(p_1 - a_1') - b_1'(p_2 - a_2') = 0$
2 PI	r_1, p_1, p_2	$Kc = -s_1 \quad Ti = -\frac{s_1 T}{s_0 + s_1}$ $T \rightarrow p_2 + r_1 p_1 - a_2' + a_1' - s_0 b_2' - s_1 b_1' = 0$
2 PD	r_1, p_1, p_2	$Kc = s_0 + s_1 \quad Ti = -\frac{s_1 T}{s_0 + s_1}$ $T \rightarrow p_2 + r_1 p_1 - a_2' - s_0 b_2' - s_1 b_1' = 0$
2 PID	$r_1, r_2,$ p_1, p_2	$Kc = -(s_1 + 2s_2)$ $Ti = \frac{(s_1 + 2s_2)T}{s_0 + s_1 + s_2} \quad Td = \frac{s_2 T}{s_1 + 2s_2}$ $T \rightarrow a_2' - a_1' + b_1' s_1 + b_2' s_0 - p_2 - (r_1 + r_2)p_1 - r_1 r_2 = 0$

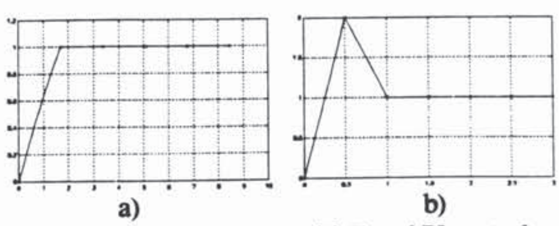
6. TUNING CRITERIA

The tuning criteria consist in placing the roots of the characteristic eq. in such a way that the specified characteristics are verified. It is supposed, that, for inputs of the step type, the system presents monotonous stable or underdamped responses. In this case the maximum number of independent characteristics that can be fixed it is equal to the number of the Gc(z) parameters. If it is used the formulas of table 7, or one more, if it is used T as an additional control parameter using formulas of table 12.

To obtain the Gc(z) parameters and T, proceed as it was showed in section 5, being the parameters of Gl(s) or Gle(s), δ, ω_n, τ_i and the following time response characteristics: rise time tr, the relation damped R, the oscillation period Ps and the maximum overshoot Mp (or others related to them). If it is used the table 12, it can be considered the number of samples between certain time instants t_i ($Nt_i=t_i/T$) as an additional characteristic. The determination of Y(t) is normally tedious, it can be fixed some characteristics by means of simulation or, in an approximate way, by means of relations obtained on simpler cases, or by using both procedures. For the systems considered, if the characteristics of time response are R, Ps, it can be obtained δ, ω_n as follows:

$$R = e^{-\frac{2\pi\delta}{\sqrt{1-\delta^2}}}, \quad Ps = \frac{2\pi}{\omega_n\sqrt{1-\delta^2}} \qquad (17)$$

One limit case of the monotonous and underdamped responses is obtained for model 1 when the specification are T and all the roots of the discrete characteristic eq. are in the origin. The system response, to one step input, will be of the form showed in Fig. 1.a, for the P controller, or, of the form indicated in Fig. 1.b, for the PI controller. In both cases it can be used the time instant t in which it overtakes the steady state as characteristic the time response, to calculate T, obtaining T=t, for P control, and T=t/2, for PI control.



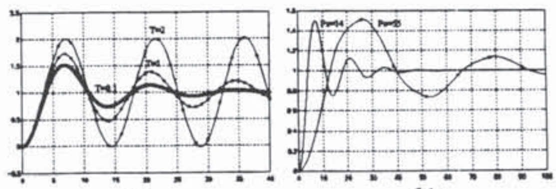
a) b)

Fig. 1. System Responses with P and PI control

7. EXAMPLE

Be Gm(s)=0.4/s(s+0.2) and the characteristics of the time response, for one unit step input, are an underdamped response, position error zero, R=1/4 and Ps according of the controller type. For the four controller type the system response are:

1.-For P controller, R=1/4 and table 7 results Ps=13.86 and Kc=0.54. If the sampling period T is reduced too enough in relation to Ps (e.g. T=0.1), the response of Fig. 2.a which verifies R=1/4 is obtained. If T is the same order as Ps, R=1/4 will not be verified as it is showed in Fig. 2.a, for T=1 and T=2. If table 12 is used, it is possible to consider two characteristics of the time responses R and Ps . The possible values of Ps can vary between that corresponding to the continuous case and the one obtained when the number of the samples until Ps is four (13.86<Ps<55.70). For R=1/4, Ps=14 and R=1/4, Ps=55 it is obtained the control parameters Kc=0.53, T=0.016 and Kc=0.06, T=12.92, respectively. The responses of the system are showed in Fig 2.b that verify the characteristics specified.

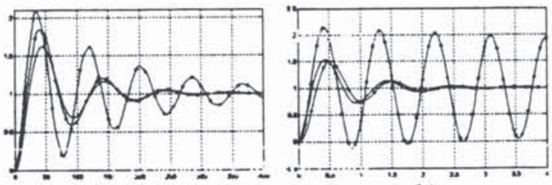


a) b)

Fig. 2. System Response with P control

2.-For PI controller and table 7, can be fixed two characteristics of the time responses. For R=1/4 and Ps=100 it is obtained the control parameters Ti=12.50, Kc=0.022 and τ_i=5.80. If the sampling period T is reduced too enough in relation to Ps, it is obtained the response showed in Fig. 3.a, which verifies the characteristics specified with Mp=63%. When T in crease the system becomes more unstable; in Fig. 3.a it is showed the response for T=15, that presents Mp>100%, R=1/2 and P=87. If used the table 12 the characteristics of the time responses similar to the continuous control with T bigger. For τ_1=15 (it will be always bigger than the continuous case), R=1/4, Ps=100, it is obtained T=16.38, Kc=0.021 and Ti=19.98, the response is showed in Fig. 3.a and it present Mp=85% and verifies the characteristics specified.

3.-For PD controller and table 7, with R=1/4, Ps=1 it is obtained Td=0.062, Kc=104, the response, for small T values in relation to Ps, is showed in Fig. 3.b, that verifies the specified characteristics with Mp=54%. When T increases the system becomes more unstable and it will not verify the specifications; in Fig. 3.b it is showed the response for T=0.1 which is almost unstable. If table 12 is used, similar characteristics are obtained to those of continuous control with T bigger. For τ_1=4 results T=0.20, Kc=3.47 and Td=3.93, the response is showed in Fig. 3.b with Mp=52%, which of verifies the characteristics specified.



a) b)

Fig. 3. System Response with PI and PD control

4.-For PID controller and table 7, can be fixed three characteristic of the time response. If R=1/4, Ps=5, Mp=50%, results τ_1=1.25, Ti=1.59, Td=0.55, Kc=5.25, the response, T=0.01, is showed in Fig. 4, that verifies the specified characteristic. When T increase the system become more unstable; in Fig. 4 it is showed the response for T=0.1, that presents Mp>100%. If it is needed responses with bigger T, making use of table 12 for τ_1=τ_2=0.2 it is obtains T=0.1635, Kc=5.1690, Ti=0.6860 and Td=1.0757, the response is in Fig. 4, that verifies the specified characteristics.

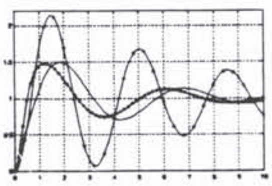


Fig. 4. System Response with PID control

8. CONCLUSIONS

1.-In the present paper is being showed how to apply the polo placement method, continuous and discrete, for the adjustment of the Gc(z) parameters and the sampling period T, in computer control for continuous processes. The tuning formulas for PID controllers and processes of first and second order have been obtained.

2.-The tuning formulas of continuous control are generally, more simple to apply than those of discrete control, but they lead to small T values that you have to adjust by means of simulation. Fast sampling can produce problems of computation time and it subjects the actuator to excessive non significatives changes. The greater complexity of the formulas of discrete control is not a disadvantage for the great computation power of present computers; on the other hand, it is possible to simplify these considering characteristics of the time response adequate.

3.-The tuning formulas of discrete control present the following advantages: 1) As the effect produced by T in the system response is included in them, they guarantee the attainment of the characteristics of time response without to having to resort to simulation. 2) The obtention of T is made as one more control parameter from the time response characteristics, and you can obtain a whole range of responses which vary from those similar to the ones of continuous control, when T tends to zero, to other types of responses. 3) The restrictions in the time response characteristics are less numerous than in the case of continuous control, because of the presence of T. 4) They permit to adjust the number of samples in relation to time response characteristics. 5) They permit to adjust T in a manner that is valid both for identification and control.

4.-When you modify T, the system changes its time response characteristics and in some cases if could become unstable. In general you can concluded that for constant values in the control parameters, when you increase the value of T the system will be faster with bigger Mp and bigger R, while if this are reduce produces the opposite effects.

9. REFERENCES

Åström, K.J. and T. Hägglund, (1988). Automatic Tuning of PID Controllers. ISA-Intrument Society of America.

Åström, K.J. and Wittenmark, B. (1990). Computer Controlled System, Theory and Design. Prentice-Hall International Editions. 2nd edition.

Dormido, S., López, I., Fernandez, J.L., Morilla F. and Aranda J. (1991). Design of Digital Control System using the Sampling Period as a Compensator. Procceding Vol. II of MELECON 91, IEEE, pp. 812-815.

Phillips, C.L. and Nagle, H.T. (1984). Digital Control Systems Analysis and Design. Englewood Cliffs, N.J.: Prentice-Hall.

Roffel, B., Vermeer, P.J. and Chin, P (1989). Simulation and Implementation of Self-Tuning Controllers. Prentice-Hall.

Shinskey, F.G. (1989). Process Control Systems: Aplication/ Desing/ Adjustment. McGraw-Hill Book Company. 3rd edition.