A ROBUST SAMPLED PI REGULATOR FOR STABLE SYSTEMS WITH MONOTONE STEP RESPONSES

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ABSTRACT

A robust discrete-time PI regulator can be designed for systems with monotone step responses, based on a simple model. The regulator has three tuning parameters; one of them is the sampling period, which can be chosen from the step response of the open loop system. Relations between the parameters assuring the stability of the system are derived.

1. Introduction

Aström [1] shows that a robust discrete-time integrating regulator can be designed for an unknown single input-single output (SISO) stable linear system based on a simplified model,

$$y(t) = bu(t-T), \ b>0.$$

For a given unknown plant with monotone step response H(t), Aström has shown that the control law

$$u_k = (r_k - y_k)/b + u_{k-1}$$

always gives a stable closed loop system provided that

$$2H(T) > H(\infty)$$

and

$$2b > H(\infty)$$

are verified, T being the sampling period, r_k the set point at time kT, $u_k = u(kT)$ the control signal, $y_k = y(kT)$ the measured output.

In Lu and Kumar [2] Aström's method is generalized by considering a staircase model for the plant. In this case the linear system is not restricted to be scalar with a monotone step response. However, the number of tuning parameters in the controller depends on the number of steps in the staircase model.

In this paper we extend Aström's approach by considering a staircase model as in Lu and Kumar, but with the magnitud of the steps decreasing in an exponential way; only one more parameter, in relation with Aström model, is introduced. (See Fig. 1). The regulator obtained has three tuning parameters, a gain, an integration time and a sampling period. Relations are found that assure asymptotic stability of the resulting closed-loop system. As in Aström, the method is constrained to systems with monotone step response; however, the introduction of one additional parameter allows a choice of smaller sampling periods; moreover, the controller now becomes of PI type so the controlled response is improved.

2. Model of the plant

Consider a stable SISO linear time-invariant plant with monotone step response H(t) as shown in Fig. 1. We consider the following model for the system

$$ym(t) = b \sum_{j=1}^{\infty} c^{j-1} u(t-jT) \qquad (1)$$

where u, ym are the input and output of the model respectivily; $b \neq 0$ has the same sign as $H(\infty)$, $0 \leq c < 1$, and T is the sampling period. Let I(t) be the unit step function, then the step response of model (1) is

$$Hm(t) = b \sum_{j=1}^{\infty} c^{j-1} I(t-jT). \qquad (2)$$

To justify the choice of model, consider a continuous model for the unknown plant given by the transfer function

$$Gm(s) = \frac{K}{s + a}.$$

The z-transform of the model preceded by a zero-order-hold gives

$$Gm(z) = \frac{b}{z - c}$$

with $b = K (1 - \exp(-aT)) / a$, $c = \exp(-aT)$. The output of the discrete system is given by

$$y(kT) = b \sum_{j=1}^{\infty} c^{j} u(kT-jT).$$

Reconstructing the unit step response of the discrete system by a zero-order interpolator gives the signal (2).

It is observed that the model suggested by Aström [1] is obtained as a special case of model (2) if we let c=0.

Let ym_k, u_k denote the output and input signals of the model at time kT. By (1) we have

$$ym_{k+1} - ym_k = bu_k + \sum_{j=1}^{\infty} (c^j - c^{j-1})\, u_{k-j} .$$

This can be rewritten in terms of the delay operator q^{-1} as

$$ym_{k+1} - ym_k = bu_k - \frac{(1-c)q^{-1}}{1 - q^{-1}} u_k$$

3. Robust sampled PI regulator

The same control law used by Aström [1] and Lu and Kumar [2] is chosen here; the model output follows the reference value r after a delay of one sampling period, so we have

$$u_k = \frac{1}{b}(r_k - ym_k) + \frac{1}{b}\,\frac{(1-c)q^{-1}}{1-q^{-1}} u_k .$$

Applying this to the real plant, the sampled control law becomes

$$u_k = \frac{1}{b}(r_k - y_k) + \frac{1 - c}{1 - q^{-1}} u_{k-1} . \qquad (3)$$

This control law is a special case of model predictive heuristic control [3-4]. It also has the form of a discrete approximation to a continuous PI strategy implemented as a simple lag in positive feedback form, Clarke [5].

4. Stability analysis

Here we follow the same reasoning as in [1]. Define $H_k=H(kT)$, $H_\infty=H(\infty)$ and let h(t) be the impulse response of the unknown plant. Then, as the control signal is constant over the sampling periods,

$$y_k = \int_0^\infty h(\tau)u(kT-\tau)\, d\tau = \sum_{j=1}^{\infty} (H_j - H_{j-1}) u_{k-j}$$

which with (3) yields

$$r_k/b = u_k + \sum_{j=1}^{\infty} (H_j - H_{j-1})/b + (c-1)c^{j-1}]u_{k-j} . \quad (4)$$

We can rewrite (4) as

$$r_k/b = \sum_{j=0}^{\infty} a_j u_{k-j}$$

where $a_0 = 1$, $a_j = (H_j - H_{j-1})/b + (c-1)c^{j-1}$.

Let us define the function $A(z) = \sum_{j=0}^{\infty} a_j z^{-j}$. It follows from Desoer and Vidyasagar (6), that the closed-loop system will be asymptotically stable if A(z) has the property

$$\inf_{|z|\geq 1} |A(z)| \geq 0.$$

Equivalently, since

$$\left| \sum_{j=1}^{\infty} a_j z^{-j} \right| \leq \sum_{j=1}^{\infty} |a_j|, \quad |z| \geq 1$$

the system is asymptotically stable if

$$\sum_{j=1}^{\infty} |a_j| < 1.$$

<u>Theorem 1.</u> Consider a stable SISO time-invariant linear system with monotone step response. The closed-loop system obtained with the regulator (3) will be asymptotically stable if the parameters b, c and the sampling period T are chosen so that the following conditions are verified

$$2(1-c) > H_\infty/b \tag{5}$$

$$(H_\infty - 2H_1)/b < 0 \tag{6}$$

Proof. See appendix 1.

Remarks.

1. If the step response is positive then conditions (5) and (6) become

$$2b(1-c) > H_\infty \tag{7}$$

$$2H_1 > H_\infty \tag{8}$$

The symbol "greater than" must be changed by the symbol "less than" in expressions (7) and (8) when the step response of the system is negative.

Condition (8) is that given in Aström [1] for the sampling period. Aström's other condition is obtained from (7) when c=0.

2. Conditions (5) and (6) gives the following design rule for the regulator (3): a) specify a sampling period T such that condition (6) is verified; b) Choose b and c such that condition (5) is verified.

The bounds given by inequalities (5) and (6) are very conservative and can be fine-tuned (see theorem 2).

<u>Theorem 2</u>. For a stable SISO time-invariant linear plant with monotone step response, the closed-loop system obtained with the regulator (3) will be asymptotically stable if the parameters b, c and the samplig period T are chosen so that either of the following conditions is true:

a)

$$c^{j-1}(1-c) \le (H_j - H_{j-1})/b, \quad j=1,..,N,M+1,....$$

$$c^{j-1}(1-c) > (H_j - H_{j-1})/b, \quad j=N+1,..,M \tag{9}$$

and

$$2(1-c^N+c^M) > (2H_N - 2H_M + H_\infty)/b; \tag{10}$$

b)

$$c^{j-1}(1-c) \ge (H_j - H_{j-1})/b, \quad j=1,..,N,M+1,....$$

$$c^{j-1}(1-c) < (H_j - H_{j-1})/b, \quad j=N+1,..,M \tag{11}$$

and

$$2(c^N - c^M) > (2H_M - 2H_N - H_\infty)/b. \qquad (12)$$

Proof. See appendix 2.

Remarks.

1. Conditions (9) and (10) may be given in terms of the increments of the sampled output at the sampling instants of both the unknown plant and the model. Let us suppose that the plant step response is positive; then b>0 and the denominator in each of (9) to (12) can be cleared. For example, condition (9) may be written as

$$bc^{j-1} - bc^{j} \le H_j - H_{j-1}$$

where bc^{j-1}, bc^{j} are the values of the step response of the model at two consecutive sampling times, and H_{j-1}, H_j are the values of step response of the plant at the same consecutive sampling times. The increments of the step response of the model of the plant decrease with time . Fig. 2 shows the values of the increments over time for a plant with positive monotone step response. Conditions (9) and (11) cover the possible relations between the unit step responses of both plant and model.

2. Both conditions in Theorem 2 a) and b) includes two important special cases i) $N=\infty$, and ii) $M=\infty$. In these cases the following four independent conditions for asymptotic stability are required:

ai)
$$c^{j-1}(1-c) \le (H_j - H_{j-1})/b, \quad j=1,2,... \qquad (13)$$

and
$$H_\infty/b > 2; \qquad (14)$$

aii)
$$c^{j-1}(1-c) \le (H_j - H_{j-1})/b, \quad j=1,..,N$$
$$c^{j-1}(1-c) > (H_j - H_{j-1})/b, \quad j=N+1,.. \qquad (15)$$

and
$$2(1-c^N) > (2H_N - H_\infty)/b; \qquad (16)$$

bi)
$$c^{j-1}(1-c) \ge (H_j - H_{j-1})/b, \quad j=1,2.. \qquad (17)$$

and
$$- H_\infty)/b < 0; \qquad (18)$$

bii)

$$c^{j-1}(1-c) \geq (H_j - H_{j-1})/b, \quad j=1,..,N$$

$$c^{j-1}(1-c) < (H_j - H_{j-1})/b, \quad j=N+1,.. \qquad (19)$$

and

$$2c^N > (H_\infty - 2H_N)/b. \qquad (20)$$

3. A PI controller may be designed based on Theorem 2. Choose a sampling period and determine the parameters b and c so that any of the conditions of the theorem is verified.

4. If the system is stable it is always possible to find b, c and T so that

$$c^{j-1}(1-c) \geq (H_j - H_{j-1})/b, \quad j=1,2.. \quad (21)$$

Then the additional condition $-H_\infty/b < 0$, assures the asymptotic stability of the system. However, this condition is implicit in the choice of b. This leads to the following corollary:

Corollary. For a stable SISO linear system with monotone step response, the closed-loop system with regulator (3) will be asymptotically stable if the parameters b, c and the sampling period T are chosen so that condition (21) is verified.

4. Examples

Consider a sixth-order plant $G(s) = 1/(1+s)^6$. According to Theorem 1, one may choose T=7.5, b=1.0 , c=0.2 so that conditions (5) and (6) are verified. Figure 3 shows the response of the closed-loop system to a step command signal at t=0.0 and to a step disturbance beginning at time t=75.

Theorem 2 allows to choose smaller sampling periods. Figure 4 shows the increments in the evolution of the positive terms of the series A(z) for T=1 ,b=1 and b=3.0, and for the negative terms when c=0.6.For b=1 condition (19) is verified with N=3, but condition (20) fails. However , with b=3.0 conditions (19) and (20) are verified with N=4. Figure 5 shows the response of the closed-loop system to a step command signal and to a step disturbance, when the values T=1, b=3. and c=0.6 are chosen.

If the sampling period is too small the number of inequalities in Theorem 2 is very long. This number may be reduced if T is increased. A reasonable value of T is five or six times smaller than the plant rise time. Figure 6 shows the response of the systems $1/(1+s)^6$, $1/(1+s)^3$ and $1/(1+s)$ to a step command signal and to a step load disturbance when T=2, T=1 and T=0.5 respectively. These values are approximately five times smaller than the rise time of the respective system; c has been fixed to 0.4. The gain of the regulator has been chosen for every system so that condition (20) is verified.

5. Conclusions

It has been shown that reasonable good control can be obtained by designing a regulator for a simplified process model. The model has been chosen so that a PI regulator results. Attemps of extending the results to the case of non-monotone systems and of developing real-time recursive algorithms for adaptive fine-tuning of the parameters are now underway.

Acknowledgment.

One of the authors, J.M. de la Cruz, would like to thank the "Comunidad Autonoma de Madrid" for a fellowship for a short stay at the Department of Engineering Science, Oxford University, where part of this work has been done.

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Appendix 1. Proof of theorem 1

Every element a_j of the series A(z), is given by the sum of a nonnegative term $(H_j - H_{j-1})/b$ and a nonpositive term $c^{j-1}(1-c)$, so it is verified

$$\sum_{j=1}^{\infty} |a_j| = |c-1+H_1/b| + \sum_{j=2}^{\infty} |a_j|$$

$$\le \left|c-1+H_1/b\right| + \sum_{j=2}^{\infty} (H_j - H_{j-1})/b + \sum_{j=2}^{\infty} c^{j-1}(1-c)$$

$$= \left|c-1+H_1/b\right| + (H_\infty - H_1)/b + c$$

Two different cases are considered sepately. First assume $H_1/b > 1-c$, then from (5)

$$\sum_{j=1}^{\infty} |a_j| \le 2c-1 + H_\infty/b < 1$$

Next, assume that $H_1/b < 1-c$, then from (6)

$$\sum_{j=1}^{\infty} |a_j| \le 1 + (H_\infty - 2H_1)/b < 1$$

which completes the proof.

Appendix 2. Proof of theorem 2

a)

$$\sum_{j=1}^{\infty} |a_j| = \sum_{j=1}^{N} [c^{j-1}(c-1) + (H_j - H_{j-1})/b] +$$

$$\sum_{j=N+1}^{M} [c^{j-1}(1-c) + (H_{j-1} - H_j)/b] +$$

$$\sum_{j=M+1}^{\infty} [c^{j-1}(c-1) + (H_j - H_{j-1})/b] =$$

$$= c^N - 1 + H_N/b + c^N - c^M + (H_N - H_M)/b - c^M +$$

$$+ (H_\infty - H_M)/b = - 1 + 2(c^N - c^M) + (2H_N - 2H_M + H_\infty)/b$$

Hence, condition (10) assures that $\sum_{j=1}^{\infty} |a_j| < 1$.

b)

$$\sum_{j=1}^{\infty} |a_j| = \sum_{j=1}^{N} [c^{j-1}(1-c) + (H_{j-1} - H_j)/b] +$$

$$\sum_{j=N+1}^{M} [c^{j-1}(c-1) + (H_j - H_{j-1})/b] +$$

$$\sum_{j=M+1}^{\infty} [c^{j-1}(1-c) + (H_{j-1} - H_j)/b] =$$

$$= 1 - c^N - H_N/b - c^N + c^M - (H_N - H_M)/b + c^M$$

$$- (H_\infty - H_M)/b = 1 - 2(c^N - c^M) - (2H_N - 2H_M + H_\infty)/b$$

Hence, condition (12) assures that $\sum_{j=1}^{\infty} |a_j| < 1$.

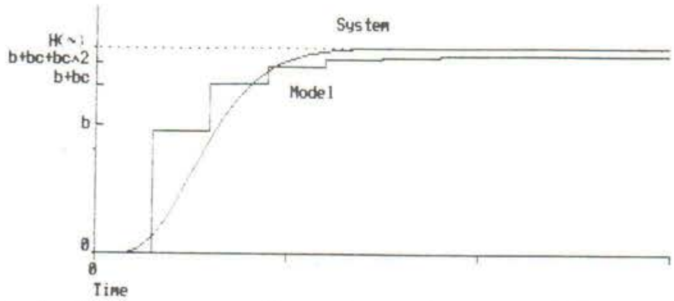


Fig. 1. Step response of the system and its approximate model.

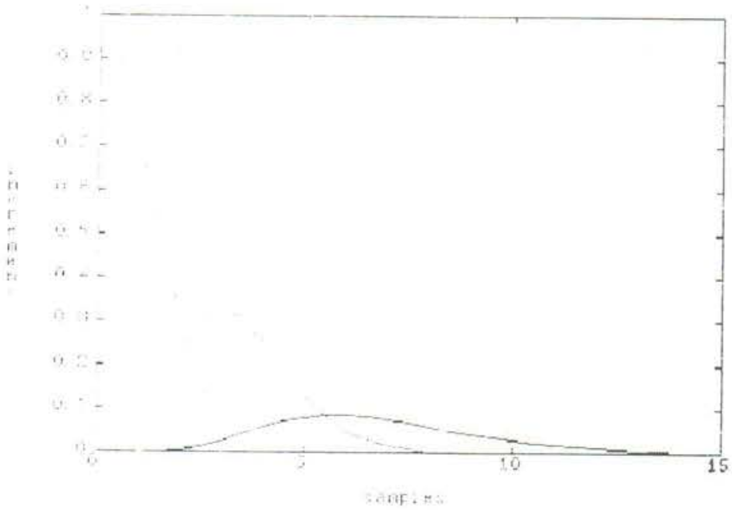


Fig. 2. Values of the increments in the output signal at the sampling instants of a monotone system, for different values of the sampling time. As the sampling time decreases the increments become smaller.

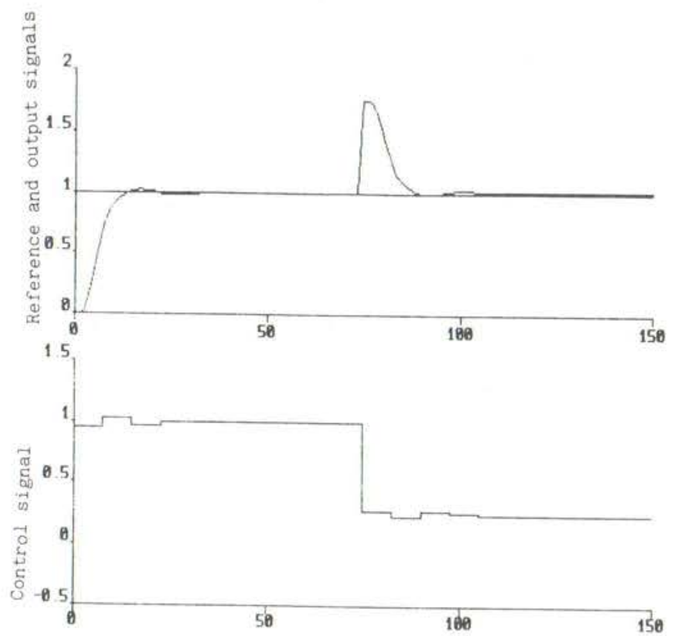


Fig. 3. Response of the system $G(s)=1/(1+s)^6$ to a step change in the reference signal and to a step load disturbance at t=75, with T=7.5, b=1.0, c=0.2.

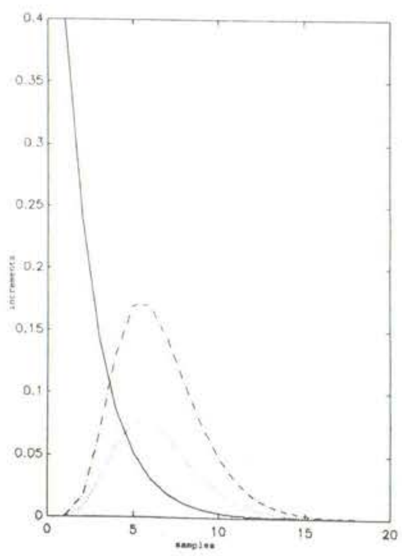


Fig. 4. ——— , c=0.6; -.-., T=1, b=1; ....,T=1, b=3.

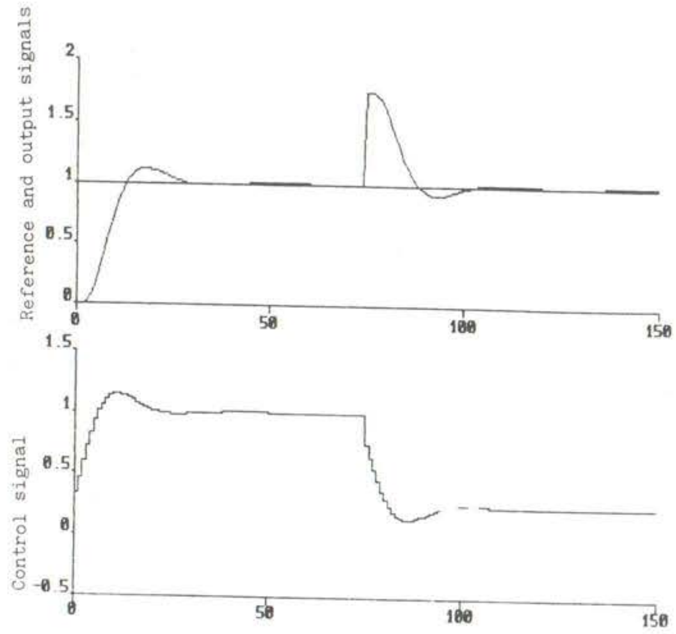
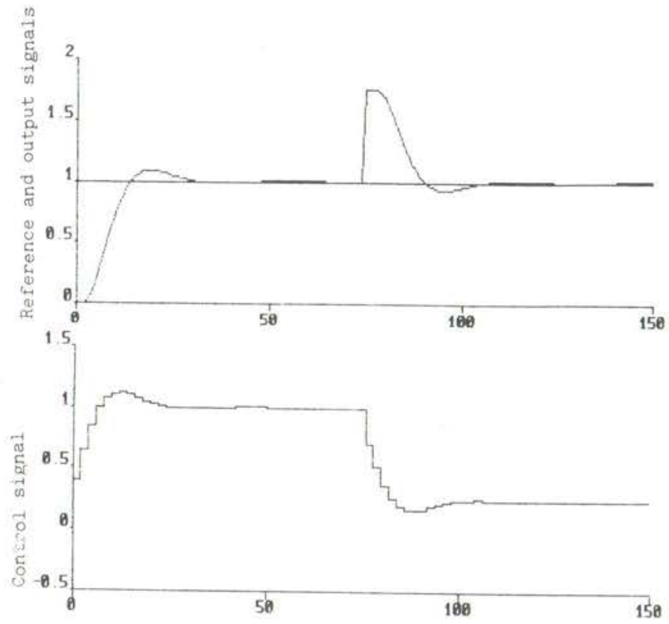
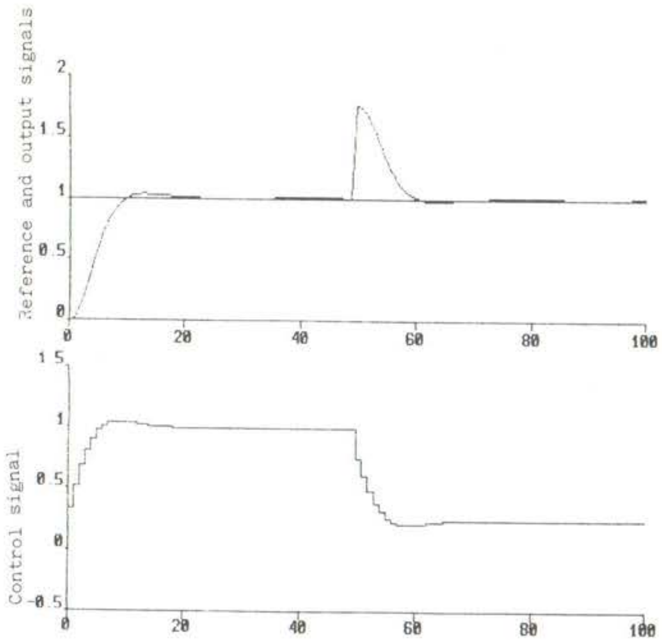


Fig. 5. Response of the system $G(s)=1/(1+s)^6$ to a step change in the reference signal and to a step load disturbance at t=75, with T=1.0, b=3.0, c=0.6.



a



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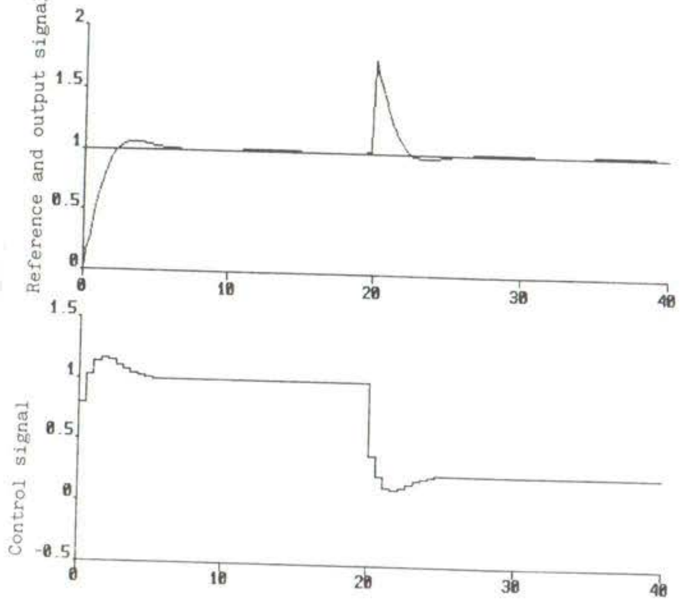

c

Fig. 6. Response to a unit step change in the reference signal and to a step load disturbance for the systems

a) $G(s)=1/(1+s)^6$, T=2 , b=2.5 , c=0.4.

b) $G(s)=1/(1+s)^3$, T=1 , b=3 , C=0.4.

c) $G(s)=1/(1+s)$, T=0.5 , b=1.25, c=0.4.