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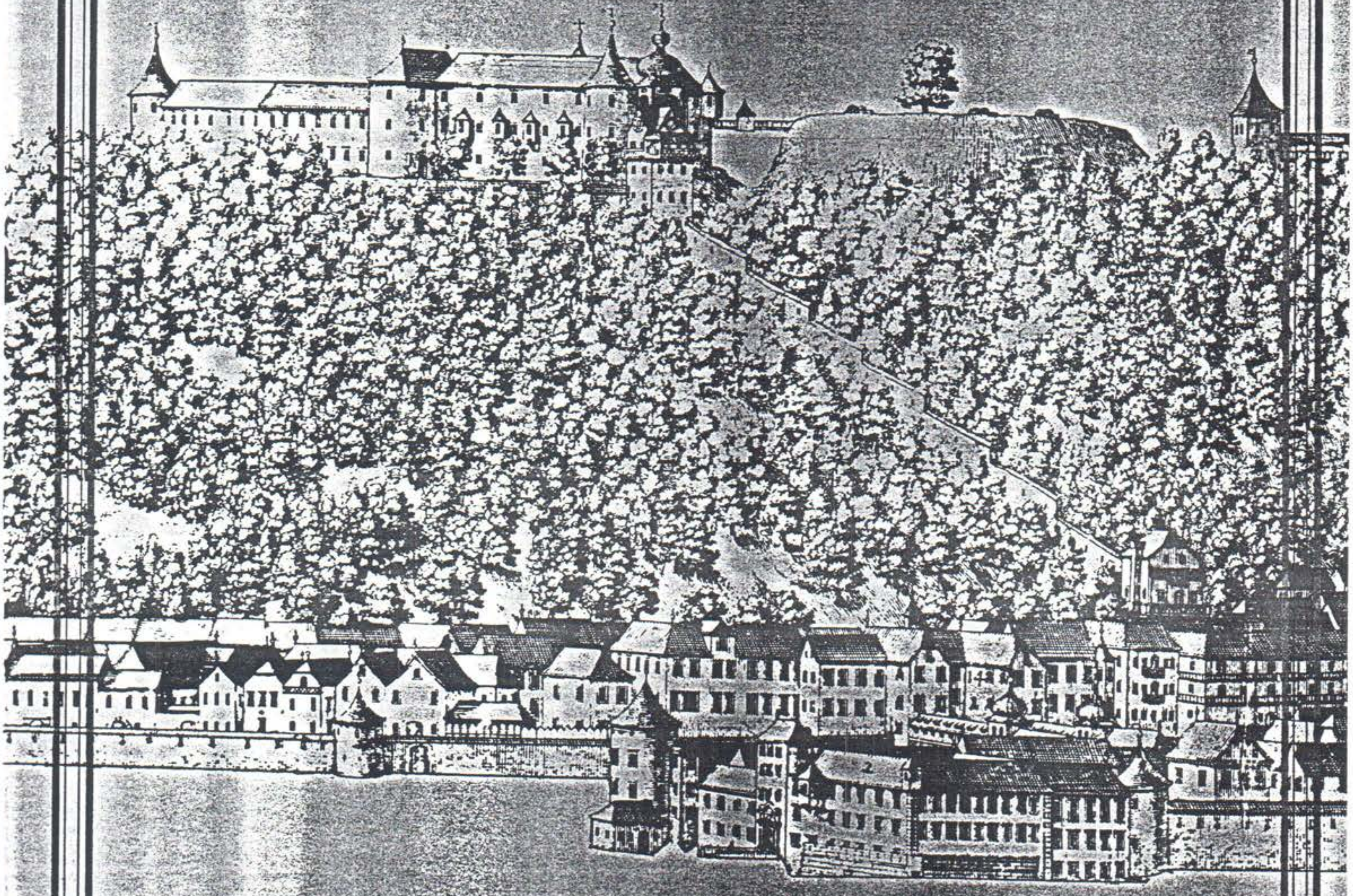

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Design of digital control system using the sampling period as a compensator

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ABSTRACT -- **The paper presents some ideas in order to use the sampling period as an additional parameter to permit the control of a digital control system. In this class of systems the sampling period becomes an important parameter to guarantee the stability and to modify widely the specification in a design problem. A modified root locus approach has been used and the sensitivity of the poles in the closed loop system is examined.**
Finally an example shows how we can compensate variation in the gain of an open loop transfer function with the sampling period to maintain constant the damping ratio.

1. Introduction

The inherent ability of a feedback control system to reduce the effect of parameter variations has long been recognized. In fact, this reduction in system sensitivity was originally employed as a quantitative measure of the advantages of feedback. The measure of sensitivity, as first introduced by Bode [1] - and expressed in the frequency domain- is still a sound tool to evaluate the performance quality of a feedback control system in the light of sensitivity requirements.
In the sixties with the emergence of some of the newer ideas in control theory the classical concept of sensitivity developed by Bode was expanded, and appeared measures of sensitivity for the closed-loop poles of a system in terms of open loop gain and of open loop poles and zeros [2],[3]. Also questions of sensitivity for sampled data system were considered by Lindorff [4]. A basic contribution was made by Horowitz [5], who showed that sensitivity analysis need not be restricted to small parameter variations. A natural consequence of this result -namely, that a conventional linear system could be designed to cope with large parameter variations - inevitably posed the question of whether (in many cases) an adaptive system is really superior to a well-designed conventional system [6].
With this point of view this paper presents a different perspective introducing the sampling period that can be considered an additional parameter in order to control the system. As a new tool we use a modified root-locus and the sensitivity of the closed loop poles when the sampling period is varied.
The paper is organized as follows. The modification of the conventional root locus so that the plot will be a function of the sampling period for a given value of gain is described in section 2.
A generalization of the results of sensitivity of the closed loop poles of Mc Ruer and Stapleford [7] is shown in section 3. Section 4 presents an example showing how we can compensate variations in the gain of an open loop transfer function varying the sampling period. Conclusions are given in section 5 and references are finally given in Section 6.

2. The modified root-locus

In the design and analysis of sampled-data feedback control systems, several analytical techniques, or a combination of analytical-graphical techniques are available. One such techniques is the root locus method. The conventional root locus method involves the determination of the movement of the roots of the characteristic equation as the gain is varied from zero to infinity.
Consider the sampled-data feedback control systems shown in figure 1. The closed-loop transfer function is given by (we have used the forward-shift operator denoted by q) :

$$\frac{C(q)}{R(q)} = \frac{D(q)\,G(q)}{1 + D(q)\,G(q)} \qquad (1)$$

The characteristic equation is simply

$$1 + D(q)\,G(q) \qquad (2)$$

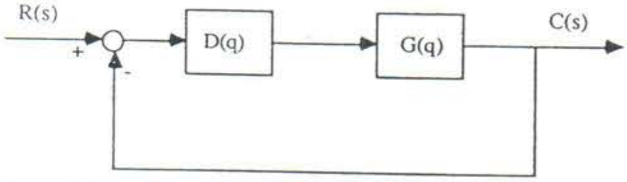

Figure 1: Closed loop sampled-data system

The conventional root locus plot is the loci of the roots of (2) when the gain, which is implicitly included, is varied from zero to infinite. This statement sounds innocent enough and in reality presents no problem if one realizes that also implicitly included in (2) is the sampling period T. Thus, in a sampled-data system, one must fix T and then the root locus method can be applied. The plot changes for each value of T chosen.
There are some systems where the importance of the sampling period is paramount to stability because of a fixed gain; that is, the gain is set and the only variable is sampling rate. It is obvious, of course, that a given system with a fixed gain may be stable at one sampling rate and unstable at another. In sampled-data system the sampling period becomes as important as the gain in stability studies. For this reason a simple technique, based on a proven existing method for determining the effect on stability of varying the sampling period has been developed. The technique is a modified root locus approach.
After studying the conventional "variable gain" root locus method, it is evident that a similar set of construction

rules for sketching the root locus of sampled-data systems as sampling period is varied and gain is held constant can be derived. An inspection of this modified root-locus plot permits the determination of the sampling period, T_{lim}, for which the system becomes unstable at a given value of gain, k.

2.1. Construction rules for the modified root-locus

The following construction rules are developed from the relation between the poles and zeros of D(q)G(q) and the roots of the characteristic equation. These rules should be regarded only as aids to the construction of the root locus plot. For an exact modified root locus plot one should probably go to the digital computer.

In most cases the construction rules for the modified root locus are the same as those for the conventional root locus. The proof for most of the rules seems unnecessary since they may be found in most textbooks on sampled-data feedback control system[8]. Where there is a difference, it has been noted.

We will go constructing a specific example. The example is taken from Åstrom and Wittenmark [9] and is the control of the double integrator, using proportional and derivative feedback(T_D). The sampling period is T and

$$D(q) = \frac{k\,(q-1)}{(q-1+k\,T_D\,T)}\quad,\quad G(q) = \frac{0.5\,T^2\,(q+1)}{(q-1)^2}$$

The characteristic equation is

$$q^2+(kT_DT-2+0.5kT^2)q+(0.5kT^2-kT_DT+1)=0 \qquad (3)$$

The roots of (3) are

$$q_{1,2} = \frac{(2-0.5\,k\,T_D\,T) \pm \Delta}{2} \qquad (4)$$

where

$$\Delta^2 = kT^2\left(0.25\,kT^2 + T_DkT + T_D^2k - 4\right) \qquad (5)$$

The problem is to determine the roots of the characteristic equation for all positive values of T and a given value of k and T_D and to plot these roots in the complex plane.

Rule 1. Starting points (T=0). The root loci start at the poles of D(q) G(q) when T=0.

Rule 2. Ending points (T=∞). The root loci end at the roots of the characteristic equation when T=∞.

As t approaches infinity, the roots of (3) are given by the following equation (we take the dominant terms in T)

$$q^2+0.5kT^2(q+1)=0 \qquad (6)$$

Rewriting (6) gives

$$\frac{1}{T2} = \frac{-0.5k(q+1))}{q^2} \qquad (7)$$

Taking limit, the ending points are q=-1 and q=-∞.

Rule 3. Number of branches of the loci. The number of branches of the loci is equal to the number of poles of the open-loop transfer function.

Since in the example the open-loop transfer function has two poles, the complete root loci will have two branches.

Rule 4. Symmetry of the root loci. The root loci are symmetrical about the real axis.

Rule 5. Root loci in the real axis. The root loci may lie on a section of the real axis if the total number of poles and zeros of D(q) G(q) to the right of the section is odd.

Rule 6. Breakaway point on the real axis. The breakaway (break-in) point on the real axis occurs at the point z_a (z_{in}) for which the sampling period T is greater (less) than for any other point on either side of z_a (z_{in}) .

The points in the complex-plane where multiple roots of the characteristic equation occur are called either the breakaway or break-in points of the root loci.

In our example the multiple roots can be obtained when $\Delta^2=0$ corresponding:

$$T = 0 \quad ,\ q = 1 \quad \text{double}$$

$$T = -2\,T_D + \frac{4}{\sqrt{k}} \quad ,\quad q = 2\,T_D\sqrt{k} - 3$$

Rule 7. Intersection of the root loci with the unit circle. The modified root loci will intersect the unit circle in the complex plane when |q| =1.0.

If the modified root loci intersects the unit circle, the limiting sampling period can be determined by substituting the value of q at the crossing point into the characteristic equation and solving for T.

In the example the result is:

$$T_{lim} = \min\left(2T_D\ ,\ \frac{2}{kT_D}\right) \qquad (8)$$

Rule 8. Calculation of T on the root loci . The value of at any point q_1 on the root loci may be determined by solving for T in the characteristic equation and substituting $q=q_1$.

Rule 9. The angle of departure (arrival) from (at) the breakaway (break-in) points. The root loci must leave the breakaway point on the real axis at an angle of θ_d, where

$$\theta_d = \frac{\left((2k+1)\,\pi + \sum \alpha_i - \sum \beta_i\right)}{N_p} \qquad (9)$$

α_i= angle from all other poles

β_i= angle from zeros

k=0,1,2,.... up to but not included the order of the characteristic equation and N_p is the number of poles at the breakaway point.

The root loci must arrive at the break-in point on the real axis at an angle of β_a apart, where

$$\beta_a = \frac{360^\circ}{N_a} \qquad (10)$$

and N_a is the number of root loci approaching the break-in point.

The modified root locus for this example is shown in figure 2 , when T_D=1.5 and k=1

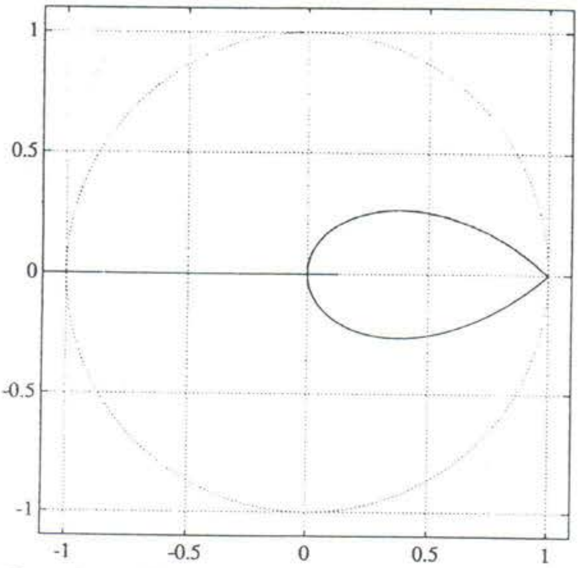

Figure 2 Modified root-locus of the example

3.Sensitivity of the closed-loop poles

In general terms, the basic problem of the sensitivity is to provide a quantitative measure of the deviation of a system function when the elements that comprise this function vary in some prescribed manner. This degree of dependence, or sensitivity, is conveniently expressed in terms of the ratio of percentage change in the function (y) to percentage change in the parameter (x)

$$S_x^y = \frac{\frac{\partial y}{y}}{\frac{\partial x}{x}} = \frac{x}{y}\frac{\partial x}{\partial y} \qquad (11)$$

For our purpose, we seek to determine the sensitivity of the closed-loop poles (q_i) of the sampled-data feedback system shown in figure 1, to variations in k_0 (open-loop gain), T (sampling period) and z_j and p_i (open-loop zeros and poles).

We will adopt the definition of sensitivity given by Mc Ruer and Stapleford [7]:

$$S_k^i = k_0\frac{\partial q_i}{\partial k_0},\quad S_{z_j}^i = k\frac{\partial q_i}{\partial z_j},\quad S_{p_k}^i = k\frac{\partial q_i}{\partial p_k} \qquad (12)$$

We also introduce the sensitivity function with respect to the sampling period T in the following manner.

$$S_T^i = \frac{\partial q_i}{\partial T} \qquad (13)$$

Definitions (12) and (13) are particularly convenient, since they satisfy a number of very useful relationships with the residues of the closed-loop transfer function. These enable the sensitivity properties of the system to be expressed in a very simple and enlightening form. We now write the open-loop transfer function as:

$$DG = DG(q, k_0, z_j(T), p_j(T)) \qquad (14)$$

to emphasize that it is a function of not only q and k_0, but also the pole and zero locations and implicitly of the sampling period T.

In the same way, q_i, is itself a function of k and T

$$q_i = q_i(\,k(k_0,T), z_j(T), p_j(T)) = q_i(k_0,T) \qquad (15)$$

We would like to compensate the deviation produced in q_i for the variation of k_0 modifying the sampling period T. The differential of q_i when T is changed to T+dT is:

$$dq_i = \sum_{j=1}^{n}\frac{\partial q_i}{\partial z_j}\frac{\partial z_j}{\partial T}dT + \sum_{j=1}^{m+n}\frac{\partial q_i}{\partial p_j}\frac{\partial p_j}{\partial T}dT + \frac{\partial q_i}{\partial k}\frac{\partial k}{\partial T}dT \qquad (16)$$

By virtue of eqs (12) and (13), the eq (16) may be written as:

$$S_T^i = \sum_{j=1}^{n} S_{z_j}^i S_T^{z_j} + \sum_{j=1}^{m+n} S_{p_j}^i S_T^{p_j} + \frac{1}{k} S_k^i S_T^k \qquad (17)$$

Recalling that

$$S_{z_j}^i = \frac{S_k^i}{(z_j - q_i)},\quad S_{p_j}^i = \frac{S_k^i}{(q_i - p_j)} \qquad (18)$$

From eqs (17) and (18) is easy to obtain

$$S_T^i = S_k^i\left[\sum_{j=1}^{n}\frac{1}{z_j - q_i}S_T^{z_j} + \sum_{j=1}^{m+n}\frac{1}{q_i - p_j}S_T^{p_j} + \frac{1}{k}S_T^k\right] \qquad (19)$$

This equation relates the sensitivity function S_T^i (the closed-loop pole q_i with respect to the sampling period T) with S_k^i (the closed-loop pole q_i with respect to the gain). The term in bracket in (19) is known.

3.1 Example:

We apply the previous result to the system analyzed in the modified root-locus (control of the double integrator), when k=1, T_d=1.5 and T=1. In this case we have a deadbeat response and the output is equal to the reference value after two samples. The closed-loop system has a double pole in zero (q_i=0).

The other parameters of the system are:

$$k = 0.5\,T^2 k_0,\; p_1 = 1 - kT_DT,\; z_1 = -1,\; p_2 = 1$$

Applying the eq (19) we deduce in this case

$$S_T^1 = -2\,S_k^1 \qquad (20)$$

This expression relates both sensitivity in a local point. If we make use of the incremental approximation, the variation Δq_i in the closed-loop pole when k_0 and T are modified in Δk and ΔT respectively, can be expressed as:

$$\Delta q_1 = S_T^1\,\Delta T + S_k^1\,\Delta k \qquad (21)$$

We try to compensate the variation in q_i produced by a modification in k, altering the sampling period T. Taking into account (20) and (21) we obtain for this example:

$$\Delta T = 0.5\,\Delta k \qquad (22)$$

4. A design example

The gain sensitivity, S_k^i (which is, in general, a complex number), has a simple physical interpretation in the complex plane. It is a vector, tangent to the root locus at $q=-q_i$ and oriented in a sense opposite to increasing k. In the same way S_T^i is a vector, tangent to the modified root locus at $q=-q_i$ and oriented in a sense opposite to increasing T.

With this consideration in mind we can use the root locus and the modified root locus to determine which modification in the sampling period can be done in order to compensate a variation in the gain of the open loop transfer function.

At this point it is instructive to consider the application of the above idea in a specific case.

The open loop transfer function of the system is given by

$$D(q)\,G(q) = \frac{k'(q+a)}{(q-1)(q-e^{-T})} \qquad (23)$$

where

$$k' = k\left(T - 1 + e^{T}\right) \quad , \quad a = \frac{1 - e^{T} - T\,e^{T}}{T - 1 + e^{T}}$$

The nominal values are $k \approx 2.98$ and $T \approx 0.17$, that correspond to a damping factor $\varphi=0.25$. It is desired that the response of the system has a constant damping factor (see figure 3)

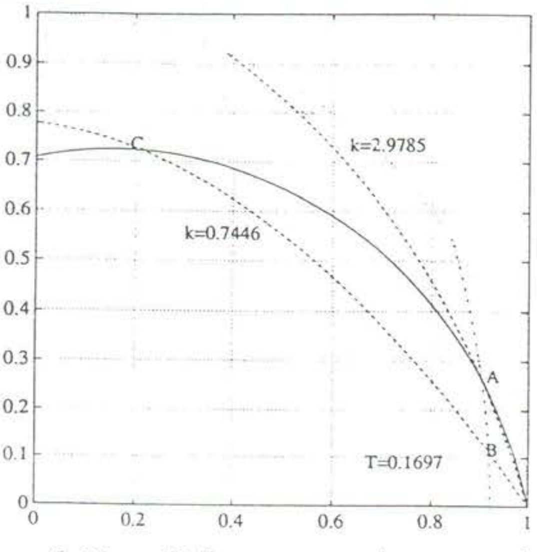

Figure 3: The solid line represents the constant damping factor locus ($\varphi=0.25$), the dashed lines the modified root locus and the dashdot line the root locus.

The point A corresponds to the nominal value, the point B is the perturbed system when the gain decrease in a 75%, and the point C the compensated system varying the sampling period in order to maintain constant the damping factor.

In figure 4 the time response is shown in the case that the gain increases in a 75%. The solid line is the response of the nominal system, the dashdot line is the response of the system perturbed in k, and the dashed line is the response of the compensated system with T.

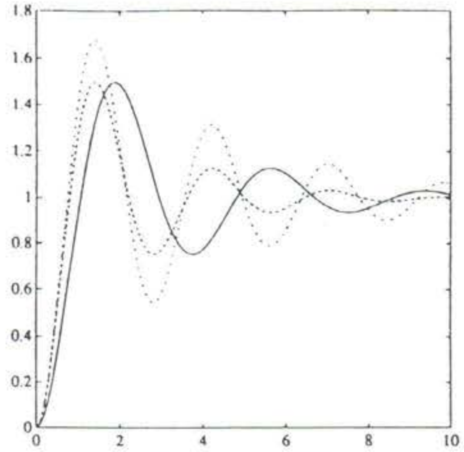

Figure 4: Response of the systems

5. Conclusions

This paper shows that it is feasible to use in a digital control system the sampling period as an additional parameter in order to compensate variations in the plant parameters. No attention has been paid in the past to this possibility.

A modified root-locus approach has been used and the sensitivity of the poles in the closed loop system is examined.

Finally an example showing how we can compensate variations in the gain of an open loop transfer function varying the sampling period is presented.

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